

# Dynamic Programming with State-Action-Dependent Discounting

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**ABSTRACT.** In this paper, we extend the discrete-time dynamic programming to the case of state-action-dependent discounting. We establish a sufficient condition known as "eventual discounting" to guarantee the standard optimality results. The condition becomes necessary for the existence of policy value given a compact state space and concave functions. Our research encompasses dynamic programming with both bounded and unbounded rewards. Furthermore, we extend the scope of eventual discounting to applications involving risk-sensitive preferences.

*Keywords:* Dynamic programming, Markov decision process, State-Action-Dependent Discounting, Optimality

## 1. INTRODUCTION

Dynamic programming or Markov decision processes (MDPs) is a robust framework for modeling and solving sequential decision-making problems. The conventional approach in dynamic programming assumes constant discount factors to capture the trade-off between present and future rewards. However, in economics and finance, discount rates vary over time (Cochrane, 2011, Hills and Nakata, 2018). The constant discounting falls short in explaining the intricacies of real-world decision-making where agents have subjective time preferences or experience exogenous uncertainty in discount rates. For example, regarding shock-dependent discounts, Justiniano and Primiceri (2008) illustrate that the variance in discount factors accounts for a significant portion of consumption volatility. Albuquerque et al. (2016) shows that risk

in time preference accounts for key asset pricing moments, such as equity premium, bond term premium, and weak correlation between stock returns and fundamentals.

Besides the uncertainty in exogenous state-dependent discounts, endogenous time preferences cannot be overlooked in economics. For instance, [Choi et al. \(2008\)](#) demonstrate that endogenously generated short-run international differences in subjective discounting, indicating increasing relative U.S. impatience, result in saving and current account imbalances that align with observed data patterns. [Hashimzade et al. \(2023\)](#) shows that a self-reinforcing redistribution mechanism, through which the endogenous discounting can lead to a higher equilibrium interest rate and a more unequal wealth distribution, in comparison to the benchmark model with a constant discount rate. [Maeda and Nagaya \(2023\)](#) show that the accelerated short-sighted consumption habit leads to earlier depletion of exhaustible resources.

This paper extends the dynamic programming theory to account for state-action-dependent discounting. We establish a comprehensive dynamic programming theory by introducing the concept of eventual discounting, where the expected multiplicative discount factors are eventually less than one for any policies. We demonstrate that eventual discounting implies the eventually contraction of Bellman operator, which is then contracting in some weighted supremum norm. Under these conditions of eventual discounting, we demonstrate the existence of an optimal policy, validate Bellman’s principle of optimality, and confirm that value function iteration, Howard policy iteration, and optimistic iteration converge to the desired value function or optimal policy.

In the existing literature, [Stachurski and Zhang \(2021\)](#) provides a complete theory of dynamic programming with state-dependent discounting, while [Sargent and Stachurski \(2023\)](#) presents a theory of dynamic programming of state-action-dependent discounting in finite state and action spaces. Moreover, [Toda \(2021\)](#) and [Toda \(2023\)](#) use the Perov Contraction Theorem to prove that the Bellman operator is contracting in some weighted supremum norm and has a unique fixed point when the exogenous state space is finite and discount factors are state-dependent.<sup>1</sup>

Addressing the gap in the literature concerning action-dependent time preferences, our exploration considers general state and action spaces with state-action-dependent

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<sup>1</sup>The weighted function is an upper bound for the reward.

discounting. In particular, we provide the standard optimality results for the dynamic programming with action-dependent discounting as outlined in [Uzawa \(1968\)](#) and [Becker and Mulligan \(1997\)](#), combining it with state-dependent stochastic discounting.

The study begins by examining the dynamic programming framework concerning bounded rewards. We show that the eventual discounting is sufficient for the existence and uniqueness of fixed points for both the Bellman operator and the policy value operators. We demonstrate that the eventually discounting condition renders these operators both eventually contracting in the supremum norm and contracting in some weighted supremum norm. This finding provides a robust foundation for the analysis of the convergence rates of these operators over time.

Furthermore, we explore the necessity of eventual discounting for the Markov decision process when state space is compact and functions are concave. We show that the condition of eventual discounting is necessary for ensuring the existence and uniqueness of policy values or the fixed points to the policy operators. In this context, we observe that the spectral radii of operators for discounted conditional expectation primarily influence the convergence rate of the system. Notably, the eigenfunctions corresponding to these spectral radii are employed as weighting vectors in weighted supremum norms.

In the extension, we generalize the result to the dynamic programming with unbounded rewards via the Q-transform, following [Ma et al. \(2022\)](#). We show that the standard optimality of dynamic programming holds when the discount factors are state-action-dependent and eventually discounting. The computation method is provided. We prove that value function iteration and action-value function iteration of the expected action-value operator converge to the optimal value or action value, which extends [Ma et al. \(2022\)](#). We also study risk-sensitive preference with eventual discounting, which ensures the uniqueness of optimal value function and the existence of optimal policy.

*Related literature.* The related literature in the Markov decision process with state- or action-dependent discount factors includes [Wei and Guo \(2011\)](#), [Minjárez-Sosa \(2015\)](#), [Wu et al. \(2015\)](#), [Wu and Zhang \(2016\)](#) and [Jasso-Fuentes et al. \(2022\)](#).

About endogenous time preferences, [Uzawa \(1968\)](#) and [Epstein and Hynes \(1983\)](#) propose a theory in which impatience rises with consumption, suggesting that the

rich are more likely to heavily discount future consumption. Although Uzawa time preference may be counterintuitive and not be supported by empirical evidence, it succeeds in theories of small open economies such as solving non-stationary issues (Schmitt-Grohé and Uribe, 2003, Guest and McDonald, 2001, Dutta and Yang, 2013). Helpman and Razin (1982) point out that when the rate of time preference remains lower than the interest rate, individuals rationally choose to accumulate foreign assets to finance their increasing consumption. The constant-discount representation of preferences cannot effectively capture this dynamic behavior. Consequently, studies on small open economies commonly employ Uzawa-type endogenous discount factors (Obstfeld, 1990, Mendoza, 1991, Schmitt-Grohé and Uribe, 2003, Durdu et al., 2009, Bodenstein, 2011, 2013, Vasilev, 2022b). For instance, Vasilev (2022a) considers Uzawa time preference, where the higher level of real income today leads to a lower discount rate, to explain the propagation of cyclical fluctuations in Bulgaria. Durdu et al. (2009) uses Uzawa endogenous time preference to model the financial globalization and the risk of Sudden Stop problem.

In contrast to Uzawa time preferences, Becker and Mulligan (1997) propose a framework that time patience is marginally increasing in future-oriented capital. Dutta and Yang (2013) establish endogenous discount factors such that marginal impatience is increasing in consumption (Uzawa type) and decreasing in future-oriented capital (Becker-Mulligan type). Their model is consistent with the empirical evidence from Australia that current consumption and turnover in future-oriented capital are positively correlated.

In empirical studies on time preferences and wealth, Lawrance (1991) reveals that the affluent exhibit greater patience than the less affluent from estimating the consumption Euler equations and the data in Panel Study of Income Dynamics. Huffman et al. (2019) examine heterogeneity in time preferences among elderly Americans and suggest that impatience correlates with lower wealth. Samwick (1998) indicates that time patience tends to increase with both income and age, by estimating the distribution of discount rates from the wealth data in Survey of Consumer Finances 1992. Cohen et al. (2020) provides a survey of the related literature.

A survey of dynamic programming with state-dependent discount rates can be found in Stachurski and Zhang (2021). In asset pricing and state-dependent discounts, since variation in asset returns is significantly due to variation in discount factors, asset pricing models consider stochastic discount factors depending on the state of

consumption growth to include adjustments for risk (Lucas Jr, 1978, Rosenberg and Engle, 2002, Cochrane, 2009, Hansen and Renault, 2010). (Campbell and Ammer, 1993, Cochrane, 2011) point out that variation in asset returns is predominantly due to variation in discount factors.

In literature of risk-sensitive preference, Hansen and Sargent (1995) develop a recursive dynamics of discounted costs for a linear quadratic exponential Gaussian linear control model, introducing risk adjustment into the framework. Moreover, Bäuerle and Jaśkiewicz (2018) studies a one-sector optimal growth model with unbounded shocks and rewards, in the framework of Hansen and Sargent (1995). They demonstrate the optimality equation for the non-expected utility and establish the Euler equation. Their analysis is based on an inequality involving associated random variables, a concept that is also utilized in the present paper. Weil (1993) develop a stochastic optimal consumption model with constant absolute risk aversion to study precautionary savings and the permanent income hypothesis. Backus et al. (2015) investigate a business cycle model incorporating aggregate risk and ambiguity. They observe that heightened uncertainty typically leads to a reduction in consumption.

If state and action spaces are finite, optimality and comparative statics can be established using fixed-point theory in complete lattices. Relevant techniques and insights can be found in works by Zhou (1994), Olszewski (2021), Balbus et al. (2022), Stachurski et al. (2022b), and Sargent and Stachurski (2023). Additionally, literature on the uniqueness of fixed-point problems for recursive preferences and dynamics includes Marinacci and Montrucchio (2010), Borovička and Stachurski (2020), Bloise et al. (2024), Ren and Stachurski (2021), and Christensen (2022).

The paper is structured as follows. Section 2 sets up the recursive decision process and presents its optimality results under eventual discounting. Section 3 presents the optimality of an eventually discounting Markov decision process. Section 4 studies the necessity of eventual discounting. Section 5 gives applications. Section 6 extends to unbounded rewards. Section 7 treats extensions in risk-sensitive preference.

## 2. RECURSIVE DECISION PROCESS WITH EVENTUAL DISCOUNTING

In this section, we introduce the framework of the recursive decision process and dynamic programming with state-action-dependent discounting. We also show the optimality results and the convergences of the conventional computation methods

under eventual discounting. The application of recursive decision processes in Markov decision processes is presented in the next section.

**2.1. Preliminary.** Let  $X$  be a metric space. Denote the family of Borel measurable (and bounded) real-valued functions on  $X$  as  $mX$  ( $mbX$ ). Denote the family of real-valued continuous (resp. upper semicontinuous) and bounded functions on  $X$  as  $cbX$  (resp.  $ubX$ ). Define  $\mathcal{B}(X)$  as the Borel  $\sigma$ -algebra on  $X$ . Given an everywhere positive function  $w$  on  $X$ , let  $\|\cdot\|_w$  be the weighted supremum norm  $\|v\|_w := \sup_{x \in X} |v(x)|/w(x)$  for all  $v \in mX$  and let  $\|\cdot\|$  be the supremum norm. Let  $U$  be a metric space. A self-map  $F$  on  $U$  is called *globally stable* if  $F$  has a unique fixed point  $u^* \in U$  and  $F^k u \rightarrow u^*$  for all  $u \in U$ . Also,  $F$  is called *eventually contracting* if there is  $k \in \mathbb{N}$  such that  $F^k$  is contracting on  $U$ . Denote  $\mathbb{1} \in mX$  as  $\mathbb{1}(x) \equiv 1$  for  $x \in X$ . Throughout,  $\leq$  is the pointwise order on  $\mathbb{R}^X$ . Denote  $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ .

**2.2. Recursive Decision Process.** The fundamental idea of dynamic programming is to solve sequential decision problems recursively through a *generic Bellman equation*

$$v(x) = \sup_{a \in \Gamma(x)} B(x, a, v) \quad (x \in X \text{ and } v \in mbX) \quad (1)$$

where

- (i)  $X$  is a Borel space, referred to as the *state space*,<sup>2</sup>
- (ii)  $A$  is a Borel space, referred to as the *action space*,
- (iii)  $\Gamma: X \rightarrow A$  is a nonempty correspondence, referred to as the *feasible correspondence*, such that  $\Gamma(x)$  is a measurable subset of  $A$  for all  $x \in X$ , which defines

- the set of the *feasible state-action pairs*

$$G := \{(x, a) \in X \times A: a \in \Gamma(x)\}, \text{ and}$$

- the set of *feasible stationary policies*

$$\Sigma := \{\sigma \in A^X: \sigma \text{ is Borel-measurable and } \sigma(x) \in \Gamma(x) \text{ for all } x \in X\},$$

- (iv) a *value aggregator*  $B: G \times mbX \rightarrow \mathbb{R}$  satisfies the monotonicity condition

$$v, w \in mbX \text{ and } v \leq w \implies B(x, a, v) \leq B(x, a, w) \quad \text{for all } (x, a) \in G. \quad (2)$$

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<sup>2</sup>A Borel space is a Borel subset of a complete and separable metric space.

A *recursive decision process* (RDP) is a tuple  $\mathcal{R} = (\mathbf{X}, \mathbf{A}, \Gamma, B)$ . We interpret  $B(x, a, v)$  as the lifetime rewards, contingent on current state  $x$  and action  $a$ , using  $v$  to evaluate future states. We assume the following regular conditions for the primitives of an RDP throughout this section.

**Condition 2.1.**

- (a)  $\Gamma$  is nonempty, continuous, and compact-valued, and
- (b)  $(x, a) \mapsto B(x, a, v)$  is bounded and measurable on  $\mathbf{G}$  for all  $v \in mb\mathbf{X}$ , and also continuous whenever  $v \in cb\mathbf{X}$ .

An RDP is called *regular* if it satisfies Condition 2.1. The regular conditions ensure that the dynamic programming is well-defined. In detail, if an RDP is regular, then the maximizer to Bellman equation (1) exists when the function  $v$  within the value aggregator is continuous.<sup>3</sup>

Given  $\sigma \in \Sigma$ , a *policy operator*  $T_\sigma: mb\mathbf{X} \rightarrow mb\mathbf{X}$  is defined by

$$T_\sigma v(x) := B(x, \sigma(x), v) \quad (x \in \mathbf{X} \text{ and } v \in mb\mathbf{X}).$$

Then,  $T_\sigma v(x)$  returns the lifetime value of action policy  $\sigma$ , under state evaluation function  $v$  at state  $x$ .

The *Bellman operator*  $T: cb\mathbf{X} \rightarrow cb\mathbf{X}$  is defined by

$$Tv(x) := \sup_{a \in \Gamma(x)} B(x, a, v) = \sup_{\sigma \in \Sigma} T_\sigma v(x) \quad (x \in \mathbf{X} \text{ and } v \in cb\mathbf{X}),$$

which is the right-hand side of (1). Then,  $Tv(x)$  returns the optimized lifetime values, under state evaluation function  $v$  at state  $x$ .

Given  $v \in mb\mathbf{X}$ , a policy  $\sigma \in \Sigma$  is called *v-greedy* if

$$\sigma(x) \in \arg \max_{a \in \Gamma(x)} B(x, a, v) \quad \text{for all } x \in \mathbf{X}.$$

A *v-greedy* policy optimizes lifetime rewards given state evaluation function  $v$ . By the definition of greedy policies,  $\sigma$  is *v-greedy* if and only if  $Tv = T_\sigma v$ .

The regular condition 2.1 guarantees the existence of a greedy policy for any  $v \in cb\mathbf{X}$ , which is a fundamental requirement in dynamic programming. The following lemma shows that whenever  $\mathcal{R}$  is regular,  $T_\sigma$  and  $T$  are well-defined on  $mb\mathbf{X}$  and  $cb\mathbf{X}$ , respectively, and *v-greedy* policies exist for all  $v \in cb\mathbf{X}$ .

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<sup>3</sup>In this paper, we call such assumptions for the existence of optimal solutions as regular conditions.

**Lemma 2.1.** *If  $\mathcal{R}$  is regular, then the following statements are true.*

- (a)  $T_\sigma$  is a self-map on  $mb\mathbf{X}$  for all  $\sigma \in \Sigma$ .
- (b)  $T$  is a self-map in  $cb\mathbf{X}$ .
- (c) For all  $v \in cb\mathbf{X}$ , there exists a  $v$ -greedy policy.

Lemma 2.1 implies that  $T_\sigma$  is a self-map for any  $\sigma \in \Sigma$  whenever  $\mathcal{R}$  is regular. It also implies that for any  $v \in cb\mathbf{X}$  there exists a  $v$ -greedy policy. Moreover, since  $T$  is a self-map on  $cb\mathbf{X}$ , it ensures that the value function iteration algorithm, introduced in Section 2.4, is well-behaved.

**2.3. Eventual Discounting.** In this section, we present the assumptions regarding stochastic state-action-dependent discounting. Analogous to the standard contracting Markov decision process that a constant discount factor is strictly less than one, we adopt the concept of eventual discounting outlined in Stachurski and Zhang (2021), Toda (2021) and Sargent and Stachurski (2023). Denote  $\rho(A)$  as the spectral radius of a bounded linear operator  $A: mb\mathbf{X} \rightarrow mb\mathbf{X}$ :

$$\rho(A) := \lim_{n \rightarrow \infty} \|A^n\|^{1/n}$$

where  $\|A\|$  denotes the operator norm of  $A$ .<sup>4</sup>

Let  $\mathcal{R} = (\mathbf{X}, \mathbf{A}, \Gamma, B)$ . Assume that there is a  $k: \mathbf{G} \times \mathbf{X} \rightarrow \mathbb{R}_+$  such that  $(x, a) \mapsto k(x, a, B)$  is a Borel measurable function on  $\mathbf{G}$  for all  $B \in \mathcal{B}(\mathbf{X})$ , and  $B \mapsto k(x, a, B)$  is a Borel measure on  $\mathbf{X}$  for all  $(x, a) \in \mathbf{G}$ . Suppose that the difference in value aggregators is bounded as follows:

$$B(x, a, v) - B(x, a, w) \leq \int_{\mathbf{X}} (v(x') - w(x')) k(x, a, dx') \quad (3)$$

for all  $(x, a) \in \mathbf{G}$  and  $v, w \in mb\mathbf{X}$ . Given  $\sigma \in \Sigma$ , let  $L_\sigma: mb\mathbf{X} \rightarrow mb\mathbf{X}$  be a positive linear operator defined by<sup>5</sup>

$$L_\sigma h(x) := \int_{\mathbf{X}} h(x') k(x, \sigma(x), dx') \quad (x \in \mathbf{X}, h \in mb\mathbf{X}). \quad (4)$$

where we denote  $L_\sigma h(x) := (L_\sigma(h))(x)$  to simplify notation. If the RDP is a Markov decision process, then  $L_\sigma h$  is interpreted as the expected discounted present value of

<sup>4</sup> $\|A\| := \inf\{c \geq 0: \|Av\| \leq c\|v\| \text{ for all } v \in mb\mathbf{X}\}.$

<sup>5</sup>A linear operator  $L$  on  $mb\mathbf{X}$  is positive if  $v \in mb\mathbf{X}$  and  $v \geq 0$  imply  $Lv \geq 0$ .



*h.* We then have<sup>6</sup>

$$\begin{aligned} |T_\sigma v(x) - T_\sigma w(x)| &\leq |B(x, \sigma(x), v) - B(x, \sigma(x), w)| \\ &\leq \int_{\mathbf{X}} |v(x') - w(x')| k(x, \sigma(x), dx') \\ &\leq L_\sigma |v - w|(x) \quad (x \in \mathbf{X}, v, w \in mb\mathbf{X}). \end{aligned}$$

We say that  $T_\sigma$  is *eventually discounting* if (3) holds and  $\rho(L_\sigma) < 1$ . Define the *eventual-discount factor*  $d_n^\sigma$  under policy  $\sigma \in \Sigma$  by

$$d_n^\sigma := \sup_{x \in \mathbf{X}} L_\sigma^n \mathbb{1}(x) \quad (n \in \mathbb{N}),$$

We will show that  $\rho(L_\sigma) < 1$  if and only if  $d_{n_\sigma}^\sigma < 1$  for some  $n_\sigma \in \mathbb{N}$ . The eventual-discount factor generalizes the constant discount factor  $\beta < 1$ . Moreover, it can be shown that  $\rho(L_\sigma) < 1$  implies the global stability of  $T_\sigma$ . To ensure that  $T$  is globally stable, we introduce the following conditions, where we consider both finite and infinite feasible policies.<sup>7</sup>

**Assumption 2.1.**

- (i) For any  $\sigma \in \Sigma$ ,  $T_\sigma$  is eventually discounting, and
- (ii)  $\Sigma$  is finite.

**Assumption 2.2.**

- (a)  $(x, a) \mapsto \int_{\mathbf{X}} h(x') k(x, a, dx')$  is continuous and bounded on  $\mathbf{G}$  for any  $h \in cb\mathbf{X}$ ,
- (b) for all  $(x, a) \in \mathbf{G}$  and  $v, w \in mb\mathbf{X}$ , (3) holds, and
- (c)  $\rho(\hat{L}) < 1$ , where  $\hat{L}: mb\mathbf{X} \rightarrow mb\mathbf{X}$  is defined by

$$\hat{L}h(x) := \sup_{\sigma \in \Sigma} L_\sigma h(x). \quad (x \in \mathbf{X}, h \in mb\mathbf{X}) \quad (5)$$

If  $\mathbf{X}$  and  $\mathbf{A}$  are finite, Assumption 2.1 is sufficient for the optimality of dynamic programming. Since Assumption 2.1 implies that each policy value  $v_\sigma$  is bounded, we can find the optimal policy from finitely many feasible policies.

If  $\Sigma$  is infinite, we consider Assumption 2.2 to ensure the global stability of  $T$ . The assumption  $\rho(\hat{L}) < 1$  implies that the Bellman operator is eventually contracting so

<sup>6</sup>We denote  $L_\sigma |v - w|(x) = L_\sigma(|v - w|)(x)$  to simplify notation.

<sup>7</sup>We sincerely thank Prof. Gaetano Bloise for pointing out the concept of monotone sublinear operators.

that we can compute the optimal value function. Moreover, [Bloise et al. \(2024\)](#) shows that  $\rho(\hat{L}) < 1$  is equivalent to that there exists an everywhere positive  $f \in mb\mathbf{X}_+$  such that for some  $\rho \in (0, 1)$  we have  $\hat{L}f \leq \rho f$ .<sup>8</sup> It implies that  $T$  is contractive in  $\|\cdot\|_f$ .

Since it may be difficult to check the eventual discounting for each policy, confirming the existence of  $n_\sigma$  satisfying  $d_{n_\sigma}^\sigma < 1$ , especially when there are infinitely many policies, we introduce the following stricter assumptions of eventual discounting.

**Assumption 2.3.** There is an  $\ell: \mathbf{X} \times \mathbf{X} \rightarrow \mathbb{R}_+$ , satisfying that  $x \mapsto \ell(x, B)$  is a Borel measurable function for all  $B \in \mathcal{B}(\mathbf{X})$ , and  $B \mapsto \ell(x, B)$  is a Borel measure on  $\mathbf{X}$  for all  $x \in \mathbf{X}$ , such that

- (i)  $L: mb\mathbf{X} \rightarrow mb\mathbf{X}$  is a positive linear operator,
- (ii)  $|B(x, a, v) - B(x, a, w)| \leq L|v - w|(x)$  for all  $x \in \mathbf{X}$ , and
- (iii)  $\rho(L) < 1$ , where

$$Lh(x) := \int_{\mathbf{X}} h(x') \ell(x, dx') \quad (x \in \mathbf{X}, h \in mb\mathbf{X}), \quad (6)$$

and  $\ell(x, x') \geq 0$  for all  $(x, x') \in \mathbf{X}^2$ .

We say that an RDP  $\mathcal{R}$  is *eventually discounting* if either Assumption 2.1, 2.2, or 2.3 holds. We provide some useful properties regarding eventual discounting assumptions. The following lemma shows the relationship between the spectral radius  $\rho(L_\sigma)$  and the expected multiplicative of eventual discount factor  $d_{n_\sigma}^\sigma$ .

**Lemma 2.2.** *If  $\mathcal{R}$  is regular, and  $L_\sigma$  and  $\hat{L}$  are defined as above, then the following statements are true.*

- (a) *For any  $\sigma \in \Sigma$ ,  $\rho(L_\sigma) < 1$  if and only if there is an  $n_\sigma \in \mathbb{N}$  such that  $d_{n_\sigma}^\sigma < 1$ .*
- (b) *For any  $\sigma \in \Sigma$ ,  $\rho(L_\sigma) = \lim_{n \rightarrow \infty} (d_n^\sigma)^{1/n}$ .*
- (c)  *$\rho(\hat{L}) < 1$  implies  $\rho(L_\sigma) < 1$  for all  $\sigma \in \Sigma$ .*

Lemma 2.2 provides a method to compute  $\rho(L_\sigma)$  by taking the limit of  $(d_n^\sigma)^{1/n}$ . Moreover, it implies that  $\rho(L_\sigma) < 1$  and  $d_{n_\sigma}^\sigma < 1$  (for some  $n_\sigma \in \mathbb{N}$ ) are equivalent. The next example with state-dependent discounting, generalized from Chapter 10 of [Stokey \(1989\)](#), is a regular RDP.

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<sup>8</sup>See Claim 6 of [Bloise et al. \(2024\)](#).

**Example 2.1** (One-Sector Optimal Growth). An economy contains many identical and infinitely lived households. There is a single good  $y_t = F(k_t, \ell_t)$  produced by capital  $k_t$  and labor  $\ell_t$  inputs. Labor is supplied inelastically such that  $\ell_t = 1$  for all  $t$ , and capital depreciates at a rate  $\delta \in (0, 1)$ . Denote  $c_t$  as consumption and  $i_t$  as investment. Given stochastic discount, a social planner solves the problem

$$\begin{aligned} & \sup \mathbb{E}_{k_0, z_0} \sum_{t=0}^{\infty} \left( \prod_{i=0}^{t-1} \beta(z_i, c_i) \right) U(c_t) \\ \text{s.t. } & c_t + i_t \leq z_t y_t, \\ & 0 \leq \ell_t \leq 1, \\ & k_{t+1} = (1 - \delta)k_t + i_t \quad (t \in \mathbb{N}_0), \\ & k_0 \geq 0 \quad \text{and} \quad z_0 \geq 0 \text{ given,} \end{aligned}$$

where  $\{z_t\}_{t \geq 0}$  is a sequence of exogenous shocks generated by a transition kernel  $Q$  on  $(Z, \mathcal{B}(Z))$  with  $Z = [1, \bar{z}]$  for  $1 < \bar{z} < \infty$ . The state is  $x = (k, z)$ , and the action is  $a = c$ . Assume that  $F$  is continuously differentiable, strictly increasing, and strictly concave with

$$\begin{aligned} F(0, \ell) &= 0, \quad F_k(k, \ell) > 0, \quad F_\ell(k, \ell) > 0, \quad (k, \ell > 0) \\ \lim_{k \rightarrow 0} F_k(k, 1) &= \infty, \quad \lim_{k \rightarrow \infty} F_k(k, 1) = 0. \end{aligned}$$

Moreover, assume that  $U$  is continuous,  $\beta$  is continuous and strictly positive, and  $Q$  satisfies the Feller property such that  $z \mapsto \int h(z')Q(z, dz')$  is bounded and continuous for all  $h \in cbZ$ . Let  $\bar{k} > 0$  be such that  $\bar{k} = \bar{z}F(\bar{k}, 1) + (1 - \delta)\bar{k}$ . Then, the set of maintainable capital stock is  $[0, \bar{k}]$ . Let the state space be  $\mathbf{X} = [0, \bar{k}] \times [1, \bar{z}]$ , the action space be  $\mathbf{A} = [0, \bar{z}F(\bar{k}, 1)]$ , the feasible correspondence be  $\Gamma(k, z) = \{c \in \mathbf{A} : 0 \leq c \leq zF(k, 1)\}$ . Hence,  $(x, a) = ((k, z), c)$ . Define the stochastic transition kernel by

$$P((k, z), c, (dk', dz')) = Q(z, dz') \mathbb{1}\{k' = (1 - \delta)k + zF(k, 1) - c\} dk'.$$

for all  $((k, z), c, (k', z')) \in \mathbf{G} \times \mathbf{X}$ . Let the value aggregator  $B$  be

$$B((k, z), c, v) = U(c) + \mathbb{E}_{(k, z)}[\beta(z, c)v(k', z')] \quad (((k, z), c, v) \in \mathbf{G} \times mb\mathbf{X}),$$

where  $(k', z')$  is generated by  $P((k, z), c, \cdot)$ . We can check that  $\mathcal{R} = (\mathbf{X}, \mathbf{A}, \Gamma, B)$  is a regular RDP.<sup>9</sup> Moreover, we have

$$\begin{aligned} |B((k, z), c, v) - B((k, z), c, w)| &= |\mathbb{E}_{(k, z)}[\beta(z, c)v(k', z')] - \mathbb{E}_{(k, z)}[\beta(z, c)w(k', z')]| \\ &\leq \mathbb{E}_{(k, z)}\beta(z, c)|v(k', z') - w(k', z')|. \end{aligned}$$

for all  $((k, z), c) \in \mathbf{G}$  and  $v, w \in mb\mathbf{X}$ . Let  $L_\sigma: mb\mathbf{X} \rightarrow mb\mathbf{X}$  be

$$L_\sigma h(k, z) = \mathbb{E}_{(k, z)}^\sigma \beta(z, \sigma(k, z))h(k', z') \quad ((k, z) \in \mathbf{X}, h \in mb\mathbf{X}),$$

where  $\mathbb{E}_{(k, z)}^\sigma$  is the expectation conditioning on  $(k_0, z_0) = (k, z)$  under transition kernel  $P((k, z), \sigma(k, z), \cdot)$  defined above. Then, we have

$$d_n^\sigma = \sup_{(k, z) \in \mathbf{X}} L_\sigma^n \mathbb{1}(k, z) = \sup_{(k, z) \in \mathbf{X}} \mathbb{E}_{(k, z)}^\sigma \prod_{t=0}^{n-1} \beta(z_t, c_t).$$

Assume  $\beta(z, c) \leq \bar{\beta}(z)$  for all  $k$  and  $c$  such that there exists an  $n \in \mathbb{N}$ :

$$\sup_{(k, z) \in \mathbf{X}} \mathbb{E}_{(k, z)}^\sigma \prod_{t=0}^{n-1} \beta(z_t, c_t) \leq \sup_{(k, z) \in \mathbf{X}} \mathbb{E}_{(k, z)} \prod_{t=0}^{n-1} \bar{\beta}(z_t) < 1. \quad (7)$$

Define  $L: mb\mathbf{X} \rightarrow mb\mathbf{X}$  by

$$Lh(k, z) = \mathbb{E}_{(k, z)} \bar{\beta}(z)h(k', z') \quad ((k, z) \in \mathbf{X}, h \in mb\mathbf{X}).$$

Hence, (7) implies  $\sup_x L^n \mathbb{1}(x) < 1$  and then  $\rho(L) < 1$ . To this end,  $\mathcal{R}$  is eventually discounting since Assumption 2.2 or 2.3 holds.  $\square$

**2.4. Dynamic Programming.** In this section, we define the optimal properties of dynamic programming and introduce dynamic programming algorithms, including value function iteration, Howard policy iteration, and optimistic policy iteration, to search for optimal policies,

Let  $\mathcal{R} = (\mathbf{X}, \mathbf{A}, \Gamma, B)$  be an RDP. Given  $\sigma \in \Sigma$ , if the policy operator  $T_\sigma$  has a unique fixed point  $v_\sigma$ , then we call  $v_\sigma$  as  $\sigma$ -value function. We show in Section 2.5 that all policy operators have unique fixed points under eventual discounting assumptions, which implies that  $\sigma$ -value exists and is unique for any  $\sigma \in \Sigma$ . The (*optimal*) *value function*  $v^*$  of  $\mathcal{R}$  is defined by

$$v^*(x) := \sup_{\sigma \in \Sigma} v_\sigma(x) \quad (x \in \mathbf{X}).$$

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<sup>9</sup>It is also a regular Markov decision process defined in Section 3.

A policy  $\sigma^* \in \Sigma$  is called *optimal* if  $v_{\sigma^*} = v^*$ ; that is, if

$$v_{\sigma^*}(x) \geq v_s \quad \text{for all } s \in \Sigma \text{ and } x \in \mathbf{X}.$$

We say  $v$  satisfies the *Bellman equations* if  $Tv = v$ ; that is  $v$  satisfies (1). We say that *Bellman's principle of optimality* holds if

$$\sigma \in \Sigma \text{ is optimal for } \mathcal{R} \iff \sigma \text{ is } v^*\text{-greedy.}$$

That is, Bellman's principle of optimality holds whenever  $v_{\sigma} = v^* \iff Tv^* = T_{\sigma}v^*$ . Bellman's principle of optimality ensures that we can find the optimal policy if we can compute  $v^*$ , and  $v^*$ -greedy policy exists.

The conventional dynamic programming algorithms are defined as follows (see, e.g., Bertsekas (2022)). Let  $\mathcal{R}$  be regular. A sequence  $\{v_k\}_{k \geq 0} \subset cb\mathbf{X}$  is called a *value function iteration* (VFI) if  $v_{k+1} = Tv_k$  for  $k \geq 0$  with any  $v_0 \in cb\mathbf{X}$ . VFI iterates the right-hand side of (1) and converges to  $v^*$  under appropriate assumptions.

For policy iteration algorithms, we introduce the following conditions to ensure the existence of greedy policies and continuity of policy values during iterations. Note that if we know the greedy policies exist and are continuous for any value function, we do not require the following conditions.<sup>10</sup>

**Condition 2.2.**

- (a)  $\Gamma$  is nonempty, continuous, compact-valued, and convex-valued,
- (b)  $(x, a) \mapsto B(x, a, v)$  is bounded and measurable on  $\mathbf{G}$  for all  $v \in mb\mathbf{X}$ , and
- (c)  $a \mapsto B(x, a, v)$  is strictly quasi-concave for  $x \in \mathbf{X}$  all  $v \in cb\mathbf{X}$ .

Define  $\Sigma_C \subset \Sigma$  as the subset of all continuous policies. Let Condition 2.2 hold. Then, the optimal policy  $\sigma^*$  is continuous. It can be shown that  $v^* = \sup_{\sigma \in \Sigma_C} v_{\sigma}$  and  $Tv = \sup_{\sigma \in \Sigma_C} T_{\sigma}v$  for all  $v \in cb\mathbf{X}$ . It then suffices to restrict the policy iterations to continuous policies.

A sequence  $\{\sigma_k\}_{k \geq 0} \subset \Sigma_C$  is called a *Howard policy iteration* (HPI) if  $\sigma_{k+1}$  is  $v_{\sigma_k}$ -greedy for  $k \geq 0$  and any  $\sigma_0 \in \Sigma_C$ . In each iteration, HPI evaluates the policy  $\sigma$  by  $v_{\sigma}$ , and if there is improvable state under  $v_{\sigma}$ , it updates policy to  $v_{\sigma}$ -greedy policy.

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<sup>10</sup>For example, state and action spaces are finite.

Therefore, the policy values from HPI increase monotonically. Define the *Howard operator*  $H: cb\mathbf{X} \rightarrow cb\mathbf{X}$  by

$$Hv = v_\sigma \text{ where } \sigma \text{ is } v\text{-greedy.}$$

The iteration,  $\{H^k v_0\}_{k \geq 0}$  with  $v_0 = v_\sigma$  for  $\sigma \in \Sigma_C$ , of the Howard operator  $H$  is an abstract version of HPI.

A sequence  $\{v_k\}_{k \geq 0} \subset cb\mathbf{X}$  is called an *optimistic policy iteration* (OPI) if, fixing  $m \in \mathbb{N}$ ,  $v_{k+1} = T_{\sigma_k}^m v_k$  where  $\sigma_k$  is  $v_k$ -greedy for  $k \geq 0$  with  $v_0 = v_\sigma$  for any  $\sigma \in \Sigma_C$ . Observe that if  $m \rightarrow \infty$ , then OPI is the same as HPI; if  $m = 1$ , OPI is the same as VFI. Therefore, OPI can be seen as a combination of HPI and VFI. Compared to HPI, OPI approximates  $v_{\sigma_k}$  by  $T_{\sigma_k}^m v_k$ . Hence, OPI can be more efficient if the calculation of  $v_{\sigma_k}$  is time-consuming, where  $v_{\sigma_k} = \lim_{n \rightarrow \infty} T_{\sigma_k}^n v_k$ , assuming that  $T_{\sigma_k}$  is globally stable.

To illustrate OPI, define optimistic policy operator  $W: cb\mathbf{X} \rightarrow cb\mathbf{X}$  by

$$Wv = T_\sigma^m v \quad (v \in cb\mathbf{X}), \quad (8)$$

where  $\sigma$  is the first policy among the list of all  $v$ -greedy policies. Then,  $\{W^k v_0\}_{k \geq 0}$  with  $v_0 = v_\sigma$  for  $\sigma \in \Sigma_C$  is an abstract version of OPI.

**2.5. Optimality.** As discussed in the introduction and the following section, we extend the existing dynamic programming theory to incorporate the cases of action-dependent discounting, including action-dependent time preferences introduced in [Uzawa \(1968\)](#) and [Becker and Mulligan \(1997\)](#). In this section, we show the optimality results, including the existence of optimal policies and Bellman's Principle of Optimality, when an RDP is eventually discounting. Moreover, the computation algorithms, including VFI, HPI, and OPI, converge to the optimal value function or optimal policies.

In the following theorem, define  $w := \sum_{n=0}^{\infty} \hat{L}^n \mathbb{1}$  for  $\|\cdot\|_w$ .

**Theorem 2.1.** *Suppose that  $\mathcal{R}$  is regular and either Assumption 2.1, 2.2 or 2.3 holds. Then, the following statements are true.*

- (a)  $T_\sigma$  is eventually contracting on  $mb\mathbf{X}$  for all  $\sigma \in \Sigma$ ,
- (b)  $T$  is contractive on  $(cb\mathbf{X}, \|\cdot\|_w)$  if Assumption 2.2 holds,
- (c)  $v^*$  is the unique solution to the Bellman equation in  $cb\mathbf{X}$ ,

- (d) *VFI converges to  $v^*$ ,*
- (e) *Bellman's principle of optimality holds,*
- (f) *HPI and OPI converge to  $\sigma^*$  if Assumption 2.1 holds,*
- (g) *Bellman's principle of optimality holds, and*
- (h) *at least one optimal (continuous) policy exists,*

*If Condition 2.2 is satisfied and Assumption 2.2 or 2.3 holds, then*

- ( $\alpha$ ) *HPI converges to  $\sigma^*$ , and*
- ( $\beta$ ) *OPI converges to  $v^*$ ,*

*where all the iterated greedy policies to HPI and OPI are continuous.*

Theorem 2.1 generalizes the conventional dynamic programming theory to the case of state-action-dependent discounting. In detail, Theorem 2.1 shows that the optimal policy exists, the Bellman's principle of optimality holds, VFI and OPI converge to  $v^*$ , and HPI converges to  $\sigma^*$ . The Bellman's principle of optimality guarantees that we can first compute  $v^*$  by VFI and find  $\sigma^*$  by

$$\sigma^*(x) = \arg \max_a B(x, a, v^*) \quad (x \in \mathbf{X}).$$

Theorem 2.1 shows that eventual discounting is sufficient for eventual contracting or global stability of  $T_\sigma$  or  $T$ . This property allows us to analyze the convergence time of VFI.

Moreover, the value function  $v^*$  is continuous. Note that if there is a non-continuous function satisfying the Bellman equation, then it must be equal to or greater than the optimal value  $v^*$ , and there is no policy  $\sigma$  that its  $\sigma$ -value  $v_\sigma$  admits that function.<sup>11</sup> Since we are interested in  $v^*$ , which is continuous when the RDP is regular, we can restrict the iteration of VFI on the continuous functions. In addition, if Condition 2.2 holds and  $v \in cb\mathbf{X}$ , then any policy  $\sigma \in \Sigma$  is dominated by some continuous policy  $\sigma_c \in \Sigma_C$  that  $T_\sigma v \leq T_{\sigma_c} v$ . To this end, it also suffices to focus on the continuous policies for HPI and OPI. Note that Condition 2.2 is not necessary if we can ensure that the continuous greedy policy exists for any  $v \in cb\mathbf{X}$ . In particular, if action and state spaces are finite, the greedy policy exists so that we do not need Condition 2.2.

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<sup>11</sup>That is, if  $Tw = w$  for some non-continuous  $w \in mb\mathbf{X}$ , then  $w \geq v^*$  and  $w > v_\sigma$  for any  $\sigma \in \Sigma$ .

**2.6. Generalized Blackwell's Condition.** The generalized Blackwell's condition for the global stability of the Bellman operator  $T$  or policy operator  $T_\sigma$  is provided as follows.

**Proposition 2.1.** *Let  $\mathcal{T}$  be an order-preserving self-map on  $U \subset mb\mathbf{X}$ . If there exists a positive linear operator  $G$  on  $mb\mathbf{X}$  such that  $\rho(G) < 1$  and*

$$\mathcal{T}(v + c) \leq \mathcal{T}v + Gc \quad \text{for all } c, v \in U \text{ with } c \geq 0,$$

*then  $\mathcal{T}$  is eventually contracting on  $U$ .*

Therefore, if we can check that, for any  $\sigma \in \Sigma$ , there exists a positive linear operator  $L_\sigma$  on  $mb\mathbf{X}$  such that

$$T_\sigma(v + c)(x) = B(x, \sigma(x), v + c) \leq B(x, \sigma(x), v) + L_\sigma c(x) = T_\sigma v(x) + L_\sigma c(x)$$

for all  $x \in \mathbf{X}$  and  $c, v \in mb\mathbf{X}$  with  $c \geq 0$ , then  $T_\sigma$  is eventually contracting and has a unique fixed point  $v_\sigma$ .

### 3. MARKOV DECISION PROCESS WITH EVENTUAL DISCOUNTING

In this section, we focus on a Markov decision process with state-action-dependent discounting and bounded rewards.

**3.1. Markov Decision Process.** A Markov decision process is an RDP such that the value aggregator is separated into reward and expected future continuing value. In detail, a *Markov decision process*  $\mathcal{M}$  is an RDP  $(\mathbf{X}, \mathbf{A}, \Gamma, B)$  such that

$$\begin{aligned} B(x, a, v) &:= \int_{\mathbf{X}} [r(x, a, x') + \beta(x, a, x')v(x')]P(x, a, dx') \\ &= \mathbb{E}_{x,a}[r(x, a, X') + \beta(x, a, X')v(X')] \end{aligned} \tag{9}$$

for all  $(x, a) \in \mathbf{G}$  and  $v \in mb\mathbf{X}$ , where

- $r: \mathbf{G} \times \mathbf{X} \rightarrow \mathbb{R}$  is a Borel measurable function, referred to as the *reward*,
- $\beta: \mathbf{G} \times \mathbf{X} \rightarrow \mathbb{R}_+$  be a Borel measurable and everywhere positive function, referred to as the *discount factors*, and
- $P: \mathbf{G} \times \mathbf{X} \rightarrow \mathbb{R}_+$  is a *stochastic kernel* on  $\mathbf{X}$  contingent on current state and action; that is,  $B \mapsto P(x, a, B)$  is a Borel probability measure on  $\mathbf{X}$  for all  $(x, a) \in \mathbf{G}$ , and  $(x, a) \mapsto P(x, a, B)$  is a measurable function on  $\mathbf{G}$  for all  $B \in \mathcal{B}(\mathbf{X})$ .



To simplify notation, denote  $\beta_\sigma, P_\sigma$  and  $r_\sigma$  as  $\beta_\sigma(x, x') := \beta(x, \sigma(x), x')$ ,  $P_\sigma(x, \cdot) := P(x, \sigma(x), \cdot)$ , and  $r_\sigma(x) := \mathbb{E}_x^\sigma r(x, \sigma(x), X')$ , respectively, for all  $x, x'$  and any  $\sigma \in \Sigma$ , where  $\mathbb{E}_x^\sigma$  denotes the expectation under  $P_\sigma$  transition kernel conditioning on  $x$ . Also, denote  $r(x, a) := \mathbb{E}_{x,a} r(x, a, X')$  for any  $(x, a) \in \mathbf{G}$ . Given an MDP  $\mathcal{M}$  and  $\sigma \in \Sigma$ , the policy operator  $T_\sigma$  following (9) becomes

$$T_\sigma v(x) := r_\sigma(x) + \int_{\mathbf{X}} \beta_\sigma(x, x') v(x') P_\sigma(x, x') dx' \quad (x \in \mathbf{X}, v \in mb\mathbf{X}).$$

If  $T_\sigma$  has a unique fixed point  $v_\sigma \in mb\mathbf{X}$ , then iteration implies  $v_\sigma = T_\sigma v_\sigma = T_\sigma^n v_\sigma$ . Letting  $n \rightarrow \infty$ , if it converges, we have

$$v_\sigma(x) := \mathbb{E}_x^\sigma \left[ \sum_{t=0}^{\infty} \left( \prod_{i=0}^{t-1} \beta_\sigma(X_i, X_{i+1}) \right) r_\sigma(X_t) \right] \quad (x \in \mathbf{X}). \quad (10)$$

where  $\prod_{i=0}^{-1} \beta_i^\sigma = 1$  by convention and  $\{X_t\}_{t \in \mathbb{N}_0}$  is a stochastic process such that  $X_0 = x$  and  $X_{t+1}$  is generated by  $P_\sigma(X_t, \cdot)$  for all  $t \in \mathbb{N}_0$ . Moreover, the Bellman operator becomes

$$Tv(x) = \sup_{a \in \Gamma(x)} \{r(x, a) + \mathbb{E}_{x,a} \beta(x, a, X') v(X')\} \quad (x \in \mathbf{X}, v \in cb\mathbf{X}).$$

We introduce the following regular conditions to an MDP.

**Condition 3.1.**

- (i)  $\Gamma$  is nonempty, compact-valued, and continuous.
- (ii)  $(x, a) \mapsto r(x, a)$  is continuous and bounded on  $\mathbf{G}$ .
- (iii)  $\beta$  is bounded and strictly positive,
- (iv)  $(x, a) \mapsto \int_{\mathbf{X}} f(y) \beta(x, a, y) P(x, a, dy)$  is bounded and continuous on  $\mathbf{G}$  for every  $f \in cb\mathbf{X}$ .

We say that an MDP  $\mathcal{M}$  is *regular* if condition 3.1 is satisfied. Condition 3.1 ensures that  $\mathcal{M}$  is a regular RDP and then the greedy policies exist by Lemma 2.1.

**Lemma 3.1.** *If an MDP  $\mathcal{M}$  such that Condition 3.1 holds, then it is a regular RDP, and the following statements are true.*

- (a)  $T_\sigma$  is a self-map on  $mb\mathbf{X}$  for all  $\sigma \in \Sigma$ .
- (b)  $T$  and  $T_\sigma$  are self-maps on  $cb\mathbf{X}$  for all  $\sigma \in \Sigma_C$ .
- (c) For all  $v \in cb\mathbf{X}$ , there exists a  $v$ -greedy policy.

To ensure the continuity of optimal policies or greedy policies, we introduce the conditions with concavity.

**Condition 3.2.**

- (i)  $\Gamma$  is nonempty, continuous, compact-valued, and convex-valued.
- (ii)  $(x, a) \mapsto r(x, a)$  is continuous and bounded on  $\mathbf{G}$ , and  $a \mapsto r(x, a)$  is strictly concave for all  $x \in \mathbf{X}$ .
- (iii)  $\beta$  is bounded and strictly positive,
- (iv)  $(x, a) \mapsto \int_{\mathbf{X}} f(y) \beta(x, a, y) P(x, a, dy)$  is bounded and continuous on  $\mathbf{G}$  for every  $f \in cb\mathbf{X}$ ,
- (v)  $a \mapsto \int_{\mathbf{X}} f(y) \beta(x, a, y) P(x, a, dy)$  is concave for all  $x \in \mathbf{X}$  and  $f \in cb\mathbf{X}$ .

Condition 3.2 guarantees that continuous  $v$ -greedy policies exist for all  $v \in cb\mathbf{X}$ , and HPI and OPI are well-behaved.

**3.2. Eventual Discounting.** Let  $\mathcal{M}$  be an MDP. To this end, (3) becomes

$$B(x, a, v) - B(x, a, w) \leq \int_{\mathbf{X}} (v(x') - w(x')) \beta(x, a, x') P(x, a, dx')$$

for all  $(x, a) \in \mathbf{G}$  and  $v, w \in mb\mathbf{X}$ . The corresponding  $L_\sigma$  is defined by

$$L_\sigma h(x) := \int_{\mathbf{X}} h(x') \beta(x, \sigma(x), x') P(x, \sigma(x), dx') = \mathbb{E}_x^\sigma \beta_\sigma(x, X') h(X') \quad (11)$$

for all  $x \in \mathbf{X}$  and  $h \in mb\mathbf{X}$ , which returns the expected discounted present value of  $h$ . We say that  $T_\sigma$  is *eventually discounting* if  $\rho(L_\sigma) < 1$ , or equivalently, there is  $n_\sigma \in \mathbb{N}$  such that

$$d_{n_\sigma}^\sigma := \sup_x \mathbb{E}_x^\sigma \prod_{t=0}^{n_\sigma-1} \beta_\sigma(X_t, X_{t+1}) < 1.$$

We can show by iteration that  $d_{n_\sigma}^\sigma = \sup_x L_\sigma^{n_\sigma} \mathbb{1}(x)$ . Define  $d_n: \mathbf{X} \rightarrow \mathbb{R}_+$  by

$$d_n(x) := \sup_{\sigma_1} \left\{ \mathbb{E}_x^{\sigma_1} \beta_{\sigma_1}(X_0, X_1) \sup_{\sigma_2} \left\{ \mathbb{E}_{X_1}^{\sigma_2} \beta_{\sigma_2}(X_1, X_2) \sup_{\sigma_3} \left\{ \cdots \right. \right. \right. \\ \left. \left. \left. \cdots \sup_{\sigma_n} \left\{ \mathbb{E}_{X_{n-1}}^{\sigma_n} \beta_{\sigma_n}(X_{n-1}, X_n) \right\} \cdots \right\} \right\} \right\}. \quad (12)$$

for all  $x \in \mathbf{X}$ . That is,  $d_1(x) = \sup_{\sigma_1} \mathbb{E}_x^{\sigma_1} \beta_{\sigma_1}(X_0, X_1)$ , and  $d_n(x) = \sup_{\sigma_n} \mathbb{E}_x^{\sigma_n} \beta_{\sigma_n}(x, X') d_{n-1}(X')$  for  $n \geq 2$  and  $x \in \mathbf{X}$ . We will show that  $\sup_x d_n(x) < 1$ , for some  $n \in \mathbb{N}$ , implies the global stability of  $T$ . Define  $\hat{L}: mb\mathbf{X} \rightarrow mb\mathbf{X}$  by

$$\hat{L}h(x) := \sup_{\sigma \in \Sigma} L_\sigma h(x) \quad (x \in \mathbf{X}, h \in mb\mathbf{X}.)$$

The operator  $\hat{L}h$  returns the highest expected discounted present value of  $h$  among all feasible policies that determine the subjective discount factors. Then, iteration yields  $d_n(x) = \hat{L}^n \mathbb{1}(x)$  for all  $x \in \mathbf{X}$ . We consider the following assumptions for eventual discounting.

**Assumption 3.1.**  $\Sigma$  is finite, and  $T_\sigma$  is eventually discounting for any  $\sigma \in \Sigma$ .

**Assumption 3.2.** Either  $\rho(\hat{L}) < 1$  or  $\sup_x d_n(x) < 1$  for some  $n \in \mathbb{N}$ .

**Assumption 3.3.** There is an  $\ell: \mathbf{X}^2 \rightarrow \mathbb{R}_+$ , satisfying that  $x \mapsto \ell(x, B)$  is a Borel measurable function for all  $B \in \mathcal{B}(\mathbf{X})$ , and  $B \mapsto \ell(x, B)$  is a Borel measure on  $\mathbf{X}$  for all  $x \in \mathbf{X}$ , such that

$$L_\sigma h \leq Lh$$

for any  $h \in mb\mathbf{X}_+$  and  $\rho(L) < 1$ , where  $L$  is defined by

$$Lh(x) = \int_{\mathbf{X}} h(x') \ell(x, dx')$$

for all  $x \in \mathbf{X}$ .

We say that  $\mathcal{M}$  is *eventually discounting* if either Assumption 3.1, 3.2, or 3.3 is satisfied. Eventual discounting implies that the expected discounted present value of the future reward is bounded above and converges to zero as the time approaches infinity, in the sense that  $L_\sigma^n r_\sigma \rightarrow 0$  as  $n \rightarrow \infty$  for all  $\sigma \in \Sigma$ . Assumption 3.3 is a sufficient condition to Assumption 3.1 and 3.2. Also,  $\rho(\hat{L}) < 1$  or  $\sup_x d_n(x) < 1$  for some  $n \in \mathbb{N}$  implies  $\rho(L_\sigma) < 1$  and that  $T_\sigma$  is eventually discounting for all  $\sigma \in \Sigma$ .

**Lemma 3.2.** *If  $\mathcal{M}$  is regular, and Assumption 3.3 holds, then Assumption 3.1 and 3.2 hold.*

**Example 3.1** (Firm valuation with stochastic interest rates). Assume that the state space  $\mathbf{X}$  is finite and follows a stochastic kernel  $P: \mathbf{X} \times \mathbf{X} \rightarrow \mathbb{R}_+$ . Suppose that the discount factors are

$$\beta_t := \frac{1}{1 + r_t} \quad (t \in \mathbb{N})$$

where  $r_t = r(X_t)$  denotes the (real) interest rates following a stochastic process. If  $\pi_t = \pi(X_t)$  is the profit at time  $t$ , then the expected present value of the firm is

$$v(x) = \mathbb{E}_x \sum_{t=0}^{\infty} \left( \prod_{i=0}^t \beta_i \right) \pi_t,$$

given current state  $X_0 = x$ . Let  $A(x, x') = P(x, x')/(1 + r(x))$  for all  $(x, x') \in \mathbf{X} \times \mathbf{X}$ . If  $\rho(A) < 1$ , then Assumption 3.3 is satisfied and  $v = \pi + Av$  (Sargent and Stachurski, 2023).  $\square$

**Example 3.2** (Uzawa Time Preferences). Mendoza (1991), Schmitt-Grohé and Uribe (2003), Vasilev (2022a,b), and Izadi and Lamsou (2022) study a small open economy where a representative household has Uzawa preference that the richer are more impatient than the poor. Uzawa preference has the merits that it stabilizes the small open economy and generates a non-degenerate distribution of wealth (Guest and McDonnald, 2001). The household chooses consumption  $c_t$  and working hours  $h_t$  to maximize the utility

$$\begin{aligned} & \mathbb{E}_0 \sum_{t=0}^{\infty} \theta_t U(c_t, h_t) \\ & U(c, h) = \frac{(c - \nu^{-1}h^\nu)^{1-\gamma}}{1-\gamma} \\ & \theta_{t+1} = b(c_t, h_t)\theta_t, \quad t \geq 0 \quad \text{with } \theta_0 = 1 \\ & b(c, h) = (1 + c - \nu^{-1}h^\nu)^{-\psi}, \end{aligned}$$

where  $\nu > 1$  is the labor supply elasticity,  $\psi > 0$  is the elasticity of discount factor with respect to component  $1 + c - \nu^{-1}h^\nu$ , and  $\gamma > 1$  measures the degree of relative risk aversion. The composite commodity  $c_t - \nu^{-1}h^\nu$  is assumed to be positive, and then the utility is bounded above.<sup>12</sup> We can see that  $b(c, h) < 1$  when  $c > \nu^{-1}h^\nu$ . Hence, the feasible correspondence is the subset of  $\{(c, h) : c > \nu^{-1}h^\nu\}$  such that the discount factors are strictly less than one. To solve the problem by discretization in programming, we can assume there is  $\varepsilon > 0$  such that  $c - \nu^{-1}h^\nu \geq \varepsilon > 0$  and then  $b(c, h) \leq (1+\varepsilon)^{-\psi} < 1$  for any  $(c, h)$ , so there exists an  $n \in \mathbb{N}$  such that  $\sup_x d_n(x) < 1$ , defined by (12), whence Assumption 3.2 or 3.3 holds.<sup>13</sup>  $\square$

<sup>12</sup>Otherwise, since  $\gamma > 1$ , if  $c \leq \nu^{-1}h^\nu$ , then  $U(c, h) \rightarrow \infty$  as  $c \uparrow \nu^{-1}h^\nu$ . Then, households can arbitrarily increase utility. Since the domain,  $c \neq \nu^{-1}h^\nu$ , is open, there is no maximizer in this case.

<sup>13</sup>On the other hand, the steady state satisfies  $\beta(c, h)(1 + r) = 1$  so that  $\beta(c, h) < 1$ , where  $r > 0$  is the real interest rate. For example, Vasilev (2022a) calibrate the parameters such that

**Example 3.3.** [Durdu et al. \(2009\)](#) consider a small open economy such that a representative household chooses the optimal consumption  $c_t$  and maximizes the preference

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \exp \left\{ - \sum_{\tau=0}^{t-1} \psi \ln(1 + c_{\tau}) \right\} \frac{c_t^{1-\gamma}}{1-\gamma},$$

where  $\psi > 0$  is the elasticity of the rate of time preference with respect to  $1 + c$ . Since  $\beta_t = \exp\{-\psi \ln(1 + c_t)\} < 1$  for  $c_t > 0$ , if we discretize the consumption space and consider inner solutions, then we have  $c \geq \varepsilon$  and  $\beta(c) \leq 1/(1 + \varepsilon)^\psi$  for some  $\varepsilon > 0$ .  $\square$

**Example 3.4** (Uzawa Time Preferences with Stochastic Discounting). This example considers both the Uzawa time preference of [Durdu et al. \(2009\)](#) and state-dependent discounting of [Hubmer et al. \(2021\)](#):<sup>14</sup>

$$\beta_t = \beta(c_t, Z_t) = Z_t \exp\{-\psi \ln(1 + c_t)\},$$

where  $c_t$  is consumption and  $Z_t$  is an AR(1) process follows

$$Z_{t+1} = \rho_Z Z_t + (1 - \rho_Z)\mu_Z + \sigma_\varepsilon \varepsilon_{t+1} \quad \{\varepsilon_t\} \stackrel{IID}{\sim} N(0, 1) \quad (13)$$

with  $\rho_Z = 0.992$ ,  $\mu_Z = 0.944$ ,  $\sigma_\varepsilon = 0.0006$ . Let  $Q$  be the transition kernel of  $Z_t$ . They discretize the process onto a grid of  $N = 15$  states by Tauchen's method which allows us to write the operator  $L$ , defined in Assumption 3.3, as a matrix

$$L_{ij} = \beta(x_i)Q(x_i, x_j), \quad 1 \leq i, j \leq N.$$

The spectral radius of matrix  $L$  is 0.9469, computed by [Stachurski and Zhang \(2021\)](#). Then, since  $\beta_t \leq Z_t$ , there exists  $n \in \mathbb{N}$  such that

$$\mathbb{E}_0 \beta_0 \beta_1 \cdots \beta_{n-1} \leq \mathbb{E}_0 Z_0 Z_1 \cdots Z_{n-1} < 1$$

for any consumption path.  $\square$

**Example 3.5** (Becker-Mulligan Time Preferences). [Becker and Mulligan \(1997\)](#) propose the endogenous time preferences that are increasing in future-oriented capital.

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$b(c, h) = 0.982$ . If we solve the model around the steady state, the discount factor is bounded away from one.

<sup>14</sup>The literature of state-dependent discounting with AR(1) process includes [Hills and Nakata \(2018\)](#), [Hills et al. \(2019\)](#) and [Nakata \(2016\)](#).

Stern (2006) considers Becker-Mulligan time preferences in an optimal growth model:

$$\begin{aligned} & \sup_{\{c_t, s_t, k_t\}_{t \geq 0}} \sum_{t=1}^{\infty} \prod_{i=1}^{t-1} \beta(s_i) U(c_t) \\ & \text{s.t. } c_t + \pi s_t + k_t \leq f(k_{t-1}), c_t \geq 0, s_t \geq 0, k_t \geq 0, \text{ for all } t \in \mathbb{N} \end{aligned}$$

where  $k_0$  is given,  $\beta: [0, \infty) \rightarrow (0, \infty)$  is continuous, concave and strictly increasing,  $\pi$  is the price of  $s_t$ , and  $k_t$  denotes capital.<sup>15</sup> Stern (2006) assumes that  $f$  is continuous and strictly increasing, and there exists a  $k_m \geq 0$  such that  $\beta(k_m/\pi) < 1$  and  $f(k) < k$  whenever  $k > k_m$ . Hence, we have  $\beta(s) \leq \beta(k_m/\pi) < 1$  for all  $s \in [0, k_m/\pi]$ , so Assumption 2.1 is satisfied.

Erol et al. (2011) also consider a similar model:

$$\begin{aligned} & \sup_{\{c_t, k_{t+1}\}_{t \geq 1}} \sum_{t=0}^{\infty} \prod_{i=1}^t \beta(k_i) U(c_t) \\ & \text{s.t. } c_t + k_{t+1} \leq f(k_t), c_t \geq 0, k_t \geq 0, \text{ for all } t \in \mathbb{N} \end{aligned}$$

where  $k_0$  is given. Erol et al. (2011) assumes that  $\beta: [0, \infty) \rightarrow (0, \infty)$  is continuous, differentiable, strictly increasing and  $\sup_{k > 0} \beta(k) < 1$ , so Assumption 3.2 is satisfied.<sup>16</sup>  $\square$

**Example 3.6** (Becker-Mulligan Preference with Stochastic Discounting). This example modifies the Becker-Mulligan Preference of Erol et al. (2011) with stochastic uncertainty. Consider a model from Example 3.5:

$$\begin{aligned} & \sup_{\{c_t, k_{t+1}\}_{t \geq 1}} \mathbb{E}_0 \sum_{t=0}^{\infty} \left( \prod_{i=1}^t Z_i \beta(k_i) \right) U(c_t) \\ & \text{s.t. } c_t + k_{t+1} \leq f(k_t), c_t \geq 0, k_t \geq 0, \text{ for all } t \in \mathbb{N} \end{aligned}$$

where  $k_0$  is given, and  $\{Z_t\}_{t \geq 0}$  is an AR(1) process satisfying that there exists  $n \in \mathbb{N}$  such that  $\mathbb{E}_0 Z_1 \cdots Z_n < 1$ . Similar to Erol et al. (2011), assumes that  $\beta: [0, \infty) \rightarrow (0, \infty)$  is continuous, differentiable, strictly increasing and  $\sup_{k > 0} \beta(k) \leq 1$ . To this end, we have

$$\mathbb{E}_0 Z_1 \beta(k_1) Z_2 \beta(k_2) \cdots Z_n \beta(k_n) \leq \mathbb{E}_0 Z_1 Z_2 \cdots Z_n < 1.$$

<sup>15</sup>The time preference  $\beta(s_t)$  is interpreted as the degree to which generation  $t$  cares for generation  $t + 1$ , while variable  $s_t$  represents actions or resources that the parent could take to strengthen the relationship with her child.

<sup>16</sup>Erol et al. (2011) also assumes that  $k$  has a compact support.

for any  $\{k_t\}_{t \geq 0}$ , which implies Assumption 3.2 and 3.3.  $\square$

**3.3. Optimality.** In this section, we present the optimality of an MDP with eventual discounting. Our results show that the standard methods to compute the optimal value and policies hold for the state-action-dependent dynamic programming such as Uzawa or Becker-Mulligan time preferences with stochastic discounting in Example 3.4 and Example 3.6.

In the following theorem, assuming eventual discounting, define

$$w_\sigma(x) := \mathbb{E}_x^\sigma \sum_{t=0}^{\infty} \prod_{i=0}^{t-1} \beta_i^\sigma \quad (x \in \mathbf{X}) \quad \text{and} \quad \lambda_\sigma := \sup_{x \in \mathbf{X}} \frac{w_\sigma(x) - 1}{w_\sigma(x)} < 1$$

Also, define

$$w := \sum_{n=1}^{\infty} \hat{L}^n \mathbb{1} \quad \text{and} \quad \lambda := \sup_{x \in \mathbf{X}} \frac{w(x) - 1}{w(x)} < 1.$$

We show in the appendix that  $L_\sigma w_\sigma \leq \lambda_\sigma w_\sigma$  for all  $\sigma \in \Sigma$  and  $\hat{L}w \leq \lambda w$ .

**Theorem 3.1.** *Let  $\mathcal{M}$  be regular. If  $\mathcal{M}$  admits  $r \equiv f \in cb\mathbf{X}_+$ , and  $T$  has a fixed point in  $cb\mathbf{X}$ , then Assumption 3.2 holds. If either Assumption 3.1, 3.2 or 3.3 holds, then the following statements are true.*

- (a)  $T_\sigma$  is eventually contracting on  $mb\mathbf{X}$  and contracting on  $(mb\mathbf{X}, \|\cdot\|_{w_\sigma})$  with modulus  $\lambda_\sigma$  for all  $\sigma \in \Sigma$ ,
- (b)  $T$  is contracting on  $(cb\mathbf{X}, \|\cdot\|_w)$  with modulus  $\lambda$  if Assumption 3.2 holds,
- (c)  $v^*$  is the unique solution to the Bellman equation in  $cb\mathbf{X}$ ,
- (d) VFI converges to  $v^*$ ,
- (e) HPI and OPI converge to  $\sigma^*$  if Assumption 3.1 holds,
- (f) Bellman's principle of optimality holds, and
- (g) at least one optimal (continuous) policy exists,

If Condition 3.2 is satisfied and Assumption 3.2 or 3.3 holds, then

- ( $\alpha$ ) HPI converges to  $\sigma^*$ , and
- ( $\beta$ ) OPI converges to  $v^*$ ,

where all the iterated greedy policies to HPI and OPI are continuous.

Theorem 3.1 generalizes the traditional constant-discounting dynamic programming theory to the case of state-action-dependent discounting. In particular, Theorem 3.1 shows that if and MDP  $\mathcal{M}$  is regular and eventually discounting, then the optimal policy exists, Bellman’s principle of optimality holds, and VFI converges to  $v^*$ . Moreover, Theorem 3.1 proves that  $T$  and  $T_\sigma$  are contractive in some weighted supremum norm. The latter immediately allows us to analyze the convergence time.

In addition, Theorem 3.1 shows that the eventual discounting condition,  $\sup_x d_n(x) < 1$  for some  $n \in \mathbf{X}$ , is necessary if we allow the reward to be only state-dependent  $r \equiv f \in cb\mathbf{X}_+$ , and there exists  $v \in cb\mathbf{X}$  satisfying the Bellman equation (i.e.,  $T$  admits a fixed point in  $cb\mathbf{X}$ .) This implies that the condition  $\rho(\hat{L}) < 1$  is a necessary condition if we want a well-behaved dynamic programming model. If  $\rho(\hat{L}) \geq 1$ , Theorem 3.1 implies that VFI does not converge, and there is no solution to the Bellman equation or the maximized lifetime value.

Moreover, Theorem 3.1 demonstrates that the optimality of Example 3.4 of Uzawa preferences or Example 3.6 of Becker-Mulligan time preferences holds true. In addition, the optimal policy can be obtained by the  $v^*$ -greedy policy, where  $v^*$  is computed by VFI.

#### 4. NECESSITY OF EVENTUAL DISCOUNTING

In this section, we investigate the necessity of eventual discounting for both the existence and uniqueness of the policy values. That is, we show the necessity of the spectral radius conditions to the existence and uniqueness of fixed points of policy operators in the environment of compact state space. Since the convergence of Howard policy iteration and optimistic policy iteration depend on the global stability of policy operators, eventual discounting for policy operators is essential to an MDP. Furthermore, we demonstrate that the convergence rate of  $T_\sigma$  is the corresponding spectral radius of the expected discounting operator.

**4.1. An MDP with a Compact State Space.** We first establish an MDP with a compact state space and positive rewards. Let  $\mathcal{M} = (\mathbf{X}, \mathbf{A}, \Gamma, B)$  be an MDP, where value aggregator  $B$  follows (9). We consider the following regular conditions. Let  $mb\mathbf{X}_+$  (resp.  $cb\mathbf{X}_+$ ) denote the set of everywhere positive functions in  $mb\mathbf{X}$  (resp.  $cb\mathbf{X}$ .) We consider the following regular conditions throughout this section.



**Condition 4.1.**

- (i)  $\mathbf{X}$  is compact,
- (ii)  $\beta$  and  $P$  are continuous, and
- (iii)  $(x, a) \mapsto r(x, a)$  is strictly positive and continuous on  $\mathbf{G}$ , and  $a \mapsto r(x, a)$  is strictly concave for all  $x \in \mathbf{X}$ ,
- (iv)  $\Gamma$  is nonempty, continuous, compact-valued and convex-valued,
- (v)  $(x, a) \mapsto \int_{\mathbf{X}} f(y) \beta(x, a, y) P(x, a, dy)$  is continuous for all  $f \in cb\mathbf{X}$ , and
- (vi)  $a \mapsto \int_{\mathbf{X}} f(y) \beta(x, a, y) P(x, a, dy)$  is concave for all  $x \in \mathbf{X}$  and  $f \in cb\mathbf{X}$ ,

Condition 4.1 is strict since it may be challenging to show that Condition 4.1 (iii) and (vi) hold at the same time. In some simple cases, Condition 4.1 (vi) holds if  $\beta$  and  $P$  are only state-dependent. Another possible condition is that  $\beta$  does not depend on future states,  $P$  is not action-dependent, and  $a \mapsto \beta(x, a) \int f(y) P(x, dy)$  is concave for all  $x \in \mathbf{X}$  and  $f \in cb\mathbf{X}$ .

Condition 4.1 is similar to Condition 3.2. The main differences are that  $\mathbf{X}$  is compact and  $r$  is strictly positive, which can be attained by scaling up a bounded reward. Then, a continuous  $v$ -greedy policy exists for any  $v \in cb\mathbf{X}$ . Moreover,  $T$  and  $T_\sigma$  are self-maps on  $cb\mathbf{X}_+$  for  $\sigma \in \Sigma_C$ .

**Lemma 4.1.** *If Condition 4.1 holds, then the following statements are true.*

- (a)  $T_\sigma$  is a self-map on  $mb\mathbf{X}_+$  for all  $\sigma \in \Sigma$ .
- (b)  $T$  and  $T_\sigma$  are self-maps on  $cb\mathbf{X}_+$  for all  $\sigma \in \Sigma_C$ .
- (c) For all  $v \in cb\mathbf{X}_+$ , there exists a continuous  $v$ -greedy policy.

An MDP  $\mathcal{M}$  is *ergodic* if for each policy  $\sigma \in \Sigma$  the induced  $P_\sigma$ -Markov chain is ergodic/irreducible. The next assumption guarantees that  $\mathcal{M}$  is ergodic. Given  $\sigma \in \Sigma$ , let  $P_\sigma^n$  be defined by  $P_\sigma^1 = P_\sigma$  and  $P_\sigma^n = \int P_\sigma(x, dy) P_\sigma^{n-1}(dy, z)$  for all  $(x, z) \in \mathbf{X} \times \mathbf{X}$  and  $i \in \mathbb{N}$ .

**Assumption 4.1.** For all  $\sigma \in \Sigma$ , there exists an  $n \in \mathbb{N}$  such that the transition density  $P_\sigma^n$  is positive everywhere.

Assumption 4.1 implies that the induced Markov chain is irreducible since  $P_\sigma^n$  is positive everywhere, and then all states will be visited eventually.

Fix  $\sigma \in \Sigma_C$ . Denote  $k_\sigma(x, dx') := \beta_\sigma(x, x')P_\sigma(x, dx')/\bar{\beta}_\sigma(x)$  for all  $(x, x') \in \mathbf{X} \times \mathbf{X}$ , where

$$\bar{\beta}_\sigma(x) := \int \beta_\sigma(x, x')P_\sigma(x, dx').$$

Note that since  $\beta$  is strictly positive,  $\bar{\beta}$  is strictly positive. We can write  $L_\sigma$  as

$$L_\sigma v(x) = \bar{\beta}_\sigma(x) \int v(x')k_\sigma(x, dx') \quad (x \in \mathbf{X}, v \in cb\mathbf{X}_+).$$

Given  $\sigma \in \Sigma_C$ , denote  $e_\sigma$  as the eigenvector of  $L_\sigma$  corresponding to its spectral radius satisfying  $L_\sigma e_\sigma = \rho(L_\sigma)e_\sigma$ . We show in the appendix that  $L_\sigma^2$  is a compact operator such that  $e_\sigma \in cb\mathbf{X}$  exists and is everywhere positive. The following proposition shows that the spectral radius and the corresponding eigenfunction dominate the contraction of the policy operator, in the sense that  $\rho(L_\sigma) < 1$  if and only if  $T_\sigma$  is contracting on  $(cb\mathbf{X}_+, \|\cdot\|_{e_\sigma})$  with modulus  $\rho(L_\sigma)$ . We consider the following eventual discounting assumption.

**Assumption 4.2.**

- (a)  $T_\sigma$  is eventually discounting for all  $\sigma \in \Sigma$ .
- (b)  $\rho(\bar{L}) < 1$ , where  $\bar{L}: cb\mathbf{X}_+ \rightarrow cb\mathbf{X}_+$  is defined by

$$\bar{L}v := \sup_{\sigma \in \Sigma_C} L_\sigma v$$

for  $v \in cb\mathbf{X}_+$ .

Given the regular conditions, we can restrict the policy set to continuous policies when we are searching for greedy policies. In this case, we focus on the sublinear operator  $\bar{L}$ . Then, the maximized expected multiplicative of discount factors,  $\bar{L}^n \mathbb{1}$ , becomes

$$d_{C,n}(x) := \sup_{\sigma_1 \in \Sigma_C} \left\{ \mathbb{E}_x^{\sigma_1} \beta_{\sigma_1}(X_0, X_1) \sup_{\sigma_2 \in \Sigma_C} \left\{ \mathbb{E}_{X_1}^{\sigma_2} \beta_{\sigma_2}(X_1, X_2) \sup_{\sigma_3 \in \Sigma_C} \left\{ \cdots \right. \right. \right. \\ \left. \left. \left. \cdots \sup_{\sigma_n \in \Sigma_C} \left\{ \mathbb{E}_{X_{n-1}}^{\sigma_n} \beta_{\sigma_n}(X_{n-1}, X_n) \right\} \cdots \right\} \right\} \right\}.$$

The eventual discounting  $\rho(\bar{L}) < 1$  is equivalent to  $\sup_x d_{C,n}(x) < 1$ . The following proposition shows that the eventual discounting of  $T_\sigma$  is necessary for the existence and uniqueness of  $v_\sigma$ .

**Proposition 4.1.** *If Assumption 4.1 and Condition 4.1 hold, then for all  $\sigma \in \Sigma_C$  the following statements are equivalent.*

- (a)  $\rho(L_\sigma) < 1$ .
- (b) *There exists an  $n_\sigma \in \mathbb{N}$  such that  $d_{n_\sigma}^\sigma < 1$ .*
- (c)  $T_\sigma$  is globally stable on  $cb\mathbf{X}_+$ .
- (d)  $T_\sigma$  is contracting on  $(cb\mathbf{X}_+, \|\cdot\|_{e_\sigma})$  with modulus  $\rho(L_\sigma)$ .
- (e)  $T_\sigma$  has a fixed point in  $cb\mathbf{X}_+$ .
- (f)  $T_\sigma$  has a unique fixed point in  $cb\mathbf{X}_+$ .

Moreover, if  $\rho(L_\sigma) \geq 1$ , then  $T_\sigma$  has no fixed point in  $cb\mathbf{X}_+$ .

Proposition 4.1 shows that  $\rho(L_\sigma) < 1$  is necessary for the existence and uniqueness of  $v_\sigma$ , the fixed point of  $T_\sigma$ . If, in addition, we have  $\sup_x d_n(x) < 1$  defined by (12), then the MDP is eventually discounting, whence the optimal properties in Theorem 3.1 are true. We summarize the results in the following theorem, which can be considered a corollary of Theorem 2.1. To apply Theorem 2.1, we use the eigenvectors as the weighting vector for the weighted supremum norm.

In the following theorem, let  $w := \sum_{n=0}^\infty \bar{L}^n \mathbb{1}$  and  $\lambda = \sup_x (w(x) - 1)/w(x)$ .

**Theorem 4.1.** *Let Condition 4.1 hold. If  $\mathcal{M}$  admits  $r \equiv f \in cb\mathbf{X}_+$ , and  $T$  has a fixed point in  $cb\mathbf{X}_+$ , then  $\rho(\bar{L}) < 1$ . If Assumption 4.1 and 4.2 hold, then the following statements are true.*

- (a)  $T_\sigma$  is eventually contracting on  $mb\mathbf{X}_+$  for all  $\sigma \in \Sigma$ , and  $T_\sigma$  is contracting with modulus  $\rho(L_\sigma)$  on  $(cb\mathbf{X}_+, \|\cdot\|_{e_\sigma})$  for all  $\sigma \in \Sigma_C$ ,
- (b)  $T$  is contracting on  $(cb\mathbf{X}_+, \|\cdot\|_w)$  with modulus  $\lambda$ ,
- (c)  $v^*$  is the unique solution to the Bellman equation in  $cb\mathbf{X}_+$ ,
- (d) HPI converges to  $\sigma^*$ ,
- (e) VFI converges to  $v^*$ ,
- (f) OPI converges to  $v^*$ ,
- (g) Bellman's principle of optimality holds, and
- (h) at least one optimal continuous policy exists.

where all the iterated greedy policies to HPI and OPI are continuous.

## 5. APPLICATIONS

In this section, we apply the main results to optimal growth and optimal default problems.

**5.1. Optimal Growth.** We can apply the main results to the following examples of optimal growth models.

**Example 5.1** (One-sector Optimal Growth, Continued). Let  $\mathcal{M}$  be the MDP defined in Example 2.1. The Bellman equation is

$$v(k, z) = \sup_{c \in [0, zF(k, 1)]} \left\{ U(c) + \beta(z, c) \int_{\mathbf{X}} v(k', z') Q(z, dz') \mathbb{1}\{k' = (1 - \delta)k + zF(k, 1) - c\} dk' \right\}.$$

Assume that there is an  $n \in \mathbb{N}$  such that

$$d_n^\sigma \leq d_n := \sup_x \sup_\sigma \mathbb{E}_x^\sigma \prod_{t=0}^{n-1} \beta(Z_t, \sigma(Z_t)) < 1$$

for all  $\sigma \in \Sigma$ . Then, we have  $\sup_{\sigma \in \Sigma_C} d_{n_\sigma}^\sigma < 1$ , so Theorem 3.1 shows that the optimality of dynamic programming holds and the related dynamic programming algorithms converge to the unique optimal value.  $\square$

**Example 5.2** (Uzawa Time Preference with Stochastic Discounting, Continued). This example continues Example 3.4 by modifying the optimal saving problem in Hubmer et al. (2021) to the case of Uzawa endogenous time preference. Following Hubmer et al. (2021), suppose that the policy operator is

$$T_\sigma v(x, z) = u(R(x, z)x + y(x, z) - \sigma(x)) + \beta(R(x, z)x + y(x, z) - \sigma(x), z) \int v(\sigma(x), z') Q(z, dz')$$

where  $x \in \mathbf{X} := \mathbb{R}_+$  is the present asset,  $z$  is the exogenous shocks generated by  $Q$  as defined by Example 3.4,  $R(x, z)$  is the gross return rates on asset holdings,  $y(x, z)$  is the labor net income,  $R(x, z)x + y(x, z) - \sigma(x) = c$  is the consumption, and  $\sigma(x)$  is the asset or saving leaving to the next period. Assume that  $\beta$  and  $\{Z_t\}$  are defined as Example 3.4. Let the utility function be

$$u(c) := \frac{c^{1-\gamma}}{1-\gamma} \quad (\gamma > 1).$$

The feasible correspondence is

$$\Gamma(x, z) := \{x' \in \mathbb{R} : \bar{x} \leq x' \leq R(x, z)x + y(x, z)\}$$

To solve the problem numerically, we discretize both  $\mathbf{X}$  and  $\mathbf{Z}$  into finite grid points. To this end, the reward function is bounded, and the continuity assumptions in Condition 3.1 are satisfied. As discussed in Example 3.4, the MDP is eventually discounting. Therefore, all of the conclusions in Theorem 3.1 hold.  $\square$

**5.2. Optimal Default.** This example considers an optimal saving problem with default following Arellano (2008), Hatchondo et al. (2009), Yue (2010), Hatchondo et al. (2016) and Ma et al. (2022). We assume state-action-dependent discounting. A country with current assets  $w_t$  chooses between continuing to participate in international financial markets and defaulting. Let  $y_t = y(Z_t, \xi_t)$  be output, where  $\{Z_t\}$  is a Markov process and  $\{\xi_t\}$  is an IID shock. Assume that default results in permanent exclusion from financial markets which yields the lifetime value

$$v^d(y, z) = \mathbb{E}_z \sum_{t=0}^{\infty} \prod_{i=0}^{t-1} \beta^d(Z_i) u(y_t),$$

where  $\beta^d(z)$  represents the stochastic discount dependent on state  $z$  given defaulting. The value of continued participation in the financial market is

$$v^c(w, y, z) = \sup_{-b \leq w' \leq R(w+y)} u(w + y - w'/R) + \beta(z, w') \mathbb{E}_z v(w', Y', Z')$$

where  $b > 0$  is a constant borrowing constraint,  $\beta(z, w')$  is the discount factor depends on state  $z$  and wealth  $w'$ , in the spirit of Becker-Mulligan time preferences in Example 3.5, and  $v$  is the value function satisfying

$$v(w, y, z) = \max\{v^d(y, z), v^c(w, y, z)\}.$$

The Bellman equation is

$$v(w, y, z) = \sup_{\substack{\delta \in \{0,1\} \\ -b \leq w' \leq R(w+y)}} \left\{ \delta \left[ \mathbb{E}_z \sum_{t=0}^{\infty} \prod_{i=0}^{t-1} \beta^d(Z_i) u(y_t) \right] + (1 - \delta) \left[ u(w + y - w'/R) + \beta(z, w') \mathbb{E}_z v(w', Y', Z') \right] \right\}.$$

Let  $\mathbf{X} = W \times Y \times Z$  where  $W$ ,  $Y$  and  $Z$  are domains of  $w_t$ ,  $y_t$  and  $z_t$ , respectively. Assume that  $u$  is continuous and either  $u$  is bounded or  $z_t$  and  $\xi_t$  have compact

supports. Suppose that  $\beta^d$  and  $\beta$  are continuous, bounded, strictly positive, that there are  $n, m \in \mathbb{N}$  such that

$$\sup_{(w,z,y) \in \mathbf{X}} \mathbb{E}_z \prod_{t=0}^{n-1} \beta^d(Z_t) < 1, \text{ and}$$

$$\sup_{\sigma \in \Sigma} \sup_{(w,z,y) \in \mathbf{X}} \mathbb{E}_z \prod_{t=0}^{m-1} \beta(Z_t, \sigma_{w'}(w_t, Y_t, Z_t)) < 1$$

where  $\sigma(w, y, z) = (\sigma_\delta(w, y, z), \sigma_{w'}(w, y, z))$  is the policy of action  $(\delta, w')$  given state  $(w, y, z) \in \mathbf{X}$ . Therefore,  $v^d$  is bounded and continuous and the MDP is eventually discounting, so all conclusions in Theorem 3.1 hold.

**5.3. Asset Pricing.** There is an ex-dividend contract that trades at prices  $\Pi_t$  and pays dividend  $D_t$ . To this end, purchasing this contract at  $t$  and selling at  $t+1$  pays  $\Pi_{t+1} + D_{t+1}$ . Let the Lucas stochastic discount factor be

$$M_{t+1} = \bar{\beta} \left( \frac{C_{t+1}}{C_t} \right)^{-\gamma},$$

where  $\bar{\beta} > 0$  is a constant discount factor measuring the impatience of the agent. Given the absence of arbitrage, the price at time  $t$  must satisfy

$$\Pi_t = \mathbb{E}_t M_{t+1} (\Pi_{t+1} + D_{t+1}). \quad (14)$$

Let  $\{X_t\}_{t \geq 0}$  be a  $Q$ -Markov process on a compact state space  $\mathbf{X}$ . Suppose that the dividend growth obeys

$$\ln \frac{D_{t+1}}{D_t} = \mu_d + X_t + \sigma_d \eta_{d,t+1}$$

where  $\{\eta_{d,t}\}_{t \geq 0}$  is IID and standard normal. Moreover, consumption growth obeys

$$\ln \frac{C_{t+1}}{C_t} = \mu_c + X_t + \sigma_c \eta_{c,t+1}$$

where  $\{\eta_{c,t}\}_{t \geq 0}$  is IID and standard normal. Let  $V_t := \Pi_t/D_t$  be the price-dividend ratio. Then, we obtain

$$\begin{aligned} V_t &= \frac{\Pi_t}{D_t} = \mathbb{E}_t \left[ M_{t+1} \frac{D_{t+1}}{D_t} \left( \frac{\Pi_{t+1}}{D_{t+1}} + 1 \right) \right] \\ &= \mathbb{E}_t \left[ \bar{\beta} \exp(-\gamma \mu_c + \mu_d + (1 - \gamma) X_t - \gamma \sigma_c \eta_{c,t+1} + \sigma_d \eta_{d,t+1}) (V_{t+1} + 1) \right]. \end{aligned} \quad (15)$$

Conditioning on  $X_t = x$ , (15) yields the value function

$$v(x) = \int_{\mathbf{X}} \bar{\beta} \exp \left( -\gamma \mu_c + \mu_d + (1 - \gamma)x + \frac{\gamma^2 \sigma_c^2 + \sigma_d^2}{2} \right) (1 + v(x')) Q(x, dx')$$

for all  $x \in \mathbf{X}$  and  $v \in cb\mathbf{X}$ . Define  $A: cb\mathbf{X} \rightarrow cb\mathbf{X}$  by

$$Af(x, x') := \int_{\mathbf{X}} \bar{\beta} \exp \left( -\gamma\mu_c + \mu_d + (1 - \gamma)x + \frac{\gamma^2\sigma_c^2 + \sigma_d^2}{2} \right) f(x') Q(x, dx')$$

for all  $x \in \mathbf{X}$  and  $f \in cb\mathbf{X}$ . We define the corresponding discount function

$$\beta(x) := \bar{\beta} \exp \left( -\gamma\mu_c + \mu_d + (1 - \gamma)x + \frac{\gamma^2\sigma_c^2 + \sigma_d^2}{2} \right)$$

for all  $x \in \mathbf{X}$ . Now, if  $x \mapsto \beta(x) \int_{\mathbf{X}} f(x') Q(x, dx')$  is continuous for any  $f \in cb\mathbf{X}$ , then a version of Theorem 3.1 without policy (or Lemma A.15) shows that  $A$  has a unique fixed point if  $\rho(A) < 1$ . Furthermore, if  $Q$  admits a continuous density and there is  $n \in \mathbb{N}$  such that  $Q^n$  is everywhere positive, then a version of Proposition 4.1 without policy shows that  $A$  has a unique fixed point if and only if  $\rho(A) < 1$ .

**5.3.1. Incomplete Market with Subjective Discounting.** This section considers the asset pricing with heterogeneous expectations in Harrison and Kreps (1978). Investors are risk-neutral. Suppose that there are finitely many investor classes, denoted by set  $\mathbf{A}$ . Agents in each class  $a \in \mathbf{A}$  have a subjective probability distribution  $P_a: \mathbf{X} \times \mathbf{X} \rightarrow \mathbb{R}_+$ . Fix the random states  $\{X_t\} \subset \mathbf{X}$ . We further assume that agents in class  $a \in \mathbf{A}$  have subjective time preferences such that  $\beta_t = \beta_a(X_t, X_{t+1})$  for  $a \in \mathbf{A}$ . Let  $\Pi_t = \pi(X_t)$  and  $D_t = d(X_t)$  for all  $t$ . Harrison and Kreps (1978) shows that the price scheme is consistent if and only if<sup>17</sup>

$$\Pi_t = \max_{a \in \mathbf{A}} \mathbb{E}^a[\beta_t(\Pi_{t+1} + D_{t+1})].$$

The Bellman equation is

$$\begin{aligned} \Pi_t &= \max_{a \in \mathbf{A}} \mathbb{E}^a[\beta_t(\Pi_{t+1} + D_{t+1})]. \\ \pi(x) &= \max_{a \in \mathbf{A}} \mathbb{E}^a[\beta_a(x, x')(\pi(x') + d(x'))] \\ &= \max_{a \in \mathbf{A}} \int_{\mathbf{X}} \beta_a(x, x')[\pi(x') + d(x')] P_a(x, dx') \end{aligned}$$

for  $x \in \mathbf{X}$ . Define  $r(x, a) := \int_{\mathbf{X}} \beta_a(x, x')\pi(x') P_a(x, dx')$  and

$$T_a\pi(x) := r(x, a) + \int_{\mathbf{X}} \beta_a(x, x')\pi(x') P_a(x, dx')$$

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<sup>17</sup>A price is called consistent when it prevents any investor from achieving an excessive expected return through adroit and legitimate speculation.

for all  $a \in \mathbf{A}$  and  $x \in \mathbf{X}$ . If  $r$  is bounded and continuous, and the MDP is eventually discounting that either Assumption 3.2 or 3.3 holds, then Theorem 3.1 shows that the optimal price  $\pi^*$  is unique, the optimality of dynamic programming holds, and the VFI will converge to  $\pi^*$ .

## 6. UNBOUNDED REWARDS

In this section, we study an eventually discounting MDP with an unbounded reward function, which generalizes the main results in Section 3.

**6.1. MDP with Unbounded Rewards.** Throughout this section, we assume the following regular condition with unbounded reward  $r: \mathbf{G} \rightarrow \mathbb{R} \cup \{-\infty, \infty\}$ .<sup>18</sup>

**Condition 6.1.**

- (i)  $\Gamma$  is nonempty, compact-valued.
- (ii)  $r$  is u.s.c..
- (iii)  $\beta$  is bounded, continuous, and strictly positive.
- (iv)  $(x, a) \mapsto \int_{\mathbf{X}} f(y) \beta(x, a, y) P(x, a, dy)$  is bounded and continuous on  $\mathbf{G}$  whenever  $f \in cb\mathbf{X}$ .

If Condition 6.1 is satisfied, define spaces  $\mathcal{V}$  and  $\mathcal{G}$  by

$$\begin{aligned} \mathcal{V} &:= \{v: \mathbf{X} \rightarrow \mathbb{R} \cup \{-\infty\}: v \text{ is u.s.c., } v/\kappa \text{ is bounded above,} \\ &\quad \text{and } (x, a) \mapsto \mathbb{E}_{x,a} v(x') \text{ is bounded below}\}, \\ \mathcal{G} &:= \{g: \mathbf{G} \rightarrow \mathbb{R}: g \text{ is u.s.c. and bounded below, } \|g\|_{\kappa} < \infty\}. \end{aligned}$$

The regular conditions, with the assumptions below, ensure the existence of a greedy policy. In this section, we say that an MDP is *regular* if Condition 6.1 holds.

Define the maximal reward function  $\bar{r}$  and (conditionally) expected maximal reward function  $\hat{r}$  as

$$\bar{r} := \sup_{a \in \Gamma(x)} r(x, a) \quad (x \in \mathbf{X}) \quad \text{and} \quad \hat{r}(x, a) := \mathbb{E}_{x,a} \bar{r}(x) \quad ((x, a) \in \mathbf{G}). \quad (16)$$

We introduce a mild assumption that  $\hat{r}$  is bounded below and  $\bar{r}$  is bounded above by some function following Ma et al. (2022).

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<sup>18</sup>The conditions for the case of measurable functions are presented in the appendix.



**Assumption 6.1.**

- (i) There exist a  $\kappa: \mathbf{X} \rightarrow [1, \infty)$  and a constant  $d \geq 0$  such that  $\bar{r}(x) \leq d\kappa(x)$  and  $a \mapsto \mathbb{E}_{x,a}\kappa(X')$  is continuous for all  $x \in \mathbf{X}$ . Moreover, if Condition 6.1 holds, then  $\kappa$  is also continuous.
- (ii) There exist constant  $n \in \mathbb{N}$  and  $\alpha \in \mathbb{R}$  and a linear operator  $L: \mathcal{V} \rightarrow \mathcal{V}$  such that  $\rho(L) < 1$ ,  $\|L\| < \infty$ ,  $\|L^n\| < 1$ ,  $\alpha \in (0, 1/\|L^n\|^{1/n})$  and

$$\mathbb{E}_{x,a}\beta(x, a, X')\kappa(X')L^t\mathbb{1}(X') \leq \alpha\kappa(x)L^{t+1}\mathbb{1}(x)$$

for all  $(x, a) \in \mathbf{G}$  and  $t \in \{0, 1, \dots, n\}$ .

- (iii)  $\hat{r}$  is bounded below.

If  $\kappa \equiv 1$  and  $\alpha = 1$ , then Assumption 6.1 (ii) is the same as Assumption 2.3. Hence, Assumption 2.3 extends the eventual contracting of Assumption 2.3. Also, Assumption 6.1 implies that the pair  $(\mathcal{G}, \|\cdot\|_\kappa)$  is a Banach space.

Recall that the Bellman equation is

$$v(x) = \sup_{a \in \Gamma(x)} \{r(x, a) + \mathbb{E}_{x,a}[\beta(x, a, X')v(X')]\}.$$

Define the *action-value function*  $g(x, a) := \mathbb{E}_{x,a}\beta(x, a, X')v(X')$  for any  $(x, a) \in \mathbf{G}$ , which is the expected (discounted) future value conditioning on  $(x, a)$ . The Bellman equation can be written as  $v(x) = \sup_{a \in \Gamma(x)} \{r(x, a) + g(x, a)\}$ . Changing  $(x, a)$  to  $(x', a')$ , multiplying  $\beta(x, a, x')$ , and taking expectation yield

$$g(x, a) = \mathbb{E}_{x,a} \left[ \beta(x, a, X') \sup_{a' \in \Gamma(X')} \{r(X', a') + g(X', a')\} \right] \quad (17)$$

for all  $(x, a) \in \mathbf{G}$ . Define the *expected value operator*  $E: \mathbb{R}^{\mathbf{X}} \rightarrow \mathbb{R}^{\mathbf{G}}$  by

$$Ev(x, a) := \mathbb{E}_{x,a}\beta(x, a, X')v(X') \quad ((x, a) \in \mathbf{G}, v \in \mathbb{R}^{\mathbf{X}}).$$

Define the *maximum value operator*  $M: \mathcal{G} \rightarrow \mathcal{V}$  by

$$Mg(x) = \sup_{a \in \Gamma(x)} \{r(x, a) + g(x, a)\} \quad (x \in \mathbf{X}, g \in \mathcal{G}).$$

We say that a policy  $\sigma \in \Sigma$  is *g-greedy* if  $r(x, \sigma(x)) + g(x, \sigma(x)) = Mg(x)$  for all  $x \in \mathbf{X}$ . Let  $g^*$  be the solution to (17). We will show that the value function is  $v^*(x) = \sup_{a \in \Gamma(x)} \{r(x, a) + g^*(x, a)\}$  for all  $x \in \mathbf{X}$ , and a policy  $\sigma \in \Sigma$  is optimal if and only if  $\sigma$  is  $g^*$ -greedy.

Define *expected value Bellman operator*  $R$  on  $\mathcal{G}$  by

$$Rg(x, a) := \mathbb{E}_{x,a} \left[ \beta(x, a, X') \sup_{a' \in \Gamma(X')} \{rX', a'\} + g(X', a') \right] \quad ((x, a) \in \mathbb{G}, g \in \mathcal{G}).$$

The *action-value function iteration* (AFI) is the iteration  $\{g_t\}_{t \geq 0} \subset \mathcal{G}$  such that  $g_{t+1} = Rg_t$  for all  $t \in \mathbb{N}_0$  with  $g_0 \in \mathcal{G}$ .

**Lemma 6.1.** *If  $\mathcal{M}$  is regular and Assumption 6.1 hold, and  $\bar{g}$  is a fixed point of  $R$ , then  $v^* = M\bar{g}$ ,  $\bar{g} = Ev^*$ , and an optimal policy exists. Moreover, the following statements are equivalent.*

- (a) a policy  $\sigma \in \Sigma$  is optimal,
- (b)  $\sigma$  is  $\bar{g}$ -greedy, and
- (c)  $\sigma$  is  $v^*$ -greedy.

**Theorem 6.1.** *If  $\mathcal{M}$  is regular and Assumption 6.1 holds, then the following statements are true.*

- (a)  $v^*$  and  $v_\sigma$  are well-defined for all  $\sigma \in \Sigma$ ,
- (b)  $R$  is eventually contracting on  $(\mathcal{G}, \|\cdot\|_\kappa)$  and  $T$  is eventually contracting on  $(\mathcal{V}, \|\cdot\|)$ ,
- (c)  $R$  admits a unique fixed point  $g^*$  in  $\mathcal{G}$  and  $T$  admits a unique fixed point  $v^*$  in  $\mathcal{V}$ ,
- (d)  $v^* = Mg^* \in \mathcal{V}$  and  $g^* = Ev^* \in \mathcal{G}$ ,
- (e) VFI converges to  $v^*$ , AFI converges to  $g^*$ ,
- (f) at least one optimal policy exists, and
- (g) a feasible policy is optimal if and only if it is  $g^*$ -greedy if and only if it is  $v^*$ -greedy.

**6.2. Application in Optimal Savings.** An agent solves an optimal savings problem with borrowing constraint:

$$\begin{aligned} & \sup \mathbb{E} \sum_{t=0}^{\infty} \prod_{i=0}^{t-1} \beta_i u(c_t) \\ \text{s.t. } & 0 \leq c_t \leq w_t \\ & w_{t+1} = R_{t+1}(w_t - c_t) + y_{t+1} \\ & (w_0, y_0) \text{ given.} \end{aligned} \tag{18}$$

Here  $\beta_t \in (0, \infty)$  is the discount factor,  $u: \mathbb{R}_+ \rightarrow \mathbb{R} \cup \{-\infty, \infty\}$  is a utility function,  $c_t$  denotes consumption,  $w_t \geq 0$  denotes wealth,  $y_t \geq 0$  denotes non-financial income, and  $R_t$  denotes the gross rate of return on financial income. Assume that asset return, income, and discount factors satisfy

$$R_t = R(\varepsilon_t), \quad y_t = y(z_t, \varepsilon_t), \quad \text{and} \quad \beta_t = \beta(z_t, z_{t+1})$$

where  $\beta$  is strictly positive, bounded and continuous,  $z_t$  is a Markov process with state space  $\mathbf{Z}$  and  $\varepsilon_t$  is an IID shock that could be vector-valued. We first note that Condition 6.1 is satisfied. The Bellman equation is

$$v(w, z) = \sup_{0 \leq c \leq w} \{u(c) + \mathbb{E}_z[\beta(z, Z')v(R(\varepsilon')(w - c) + y(Z', \varepsilon'), Z')]\}$$

To this end, the state is  $x = (w, z) \in \mathbb{R}_+ \times \mathbf{Z}$  and the action is  $a = c \in \Gamma(x) = [0, w]$ . Suppose that  $u$  is u.s.c., increasing,  $\inf_z \mathbb{E}_z u(y(Z', \varepsilon')) > -\infty$ , and there exist  $p > 0$  and  $q > 1$  such that

$$u(c) \leq pc + q \text{ for all } c > 0.$$

Define the weighting function by  $\kappa(x) = pw + q$  for all  $x = (w, z)$  and set  $d = 1$  for Assumption 6.1. To ensure Assumption 6.1, assume

$$\sup_z \mathbb{E}_z \beta(z, Z') y(Z', \varepsilon') < \infty, \quad \text{and} \quad \mathbb{E} R(\varepsilon') > 1, \quad (19)$$

In addition, assume that there is  $m \in \mathbb{N}$  such that

$$\sup_{z \in \mathbf{Z}} \mathbb{E}_z \beta_0 \beta_1 \beta_2 \cdots \beta_{m-1} (\mathbb{E} R(\varepsilon'))^m < 1.$$

This assumption is the eventual discounting for Assumption 6.1 (b). Define operator  $L$  by  $Lh(x) := \mathbb{E}_z \beta(z, Z') h(X') \mathbb{E} R(\varepsilon')$  for all  $x \in \mathbf{X}$ .

**Lemma 6.2.** *If  $\mathcal{M}$  follows (18) and all the above corresponding assumptions hold, then there is large enough  $q > 1$  such that  $\kappa(x) = pw + q$  and*

$$\mathbb{E}_{x,a} \beta(x, a, X') \kappa(X') L^n \mathbb{1}(x') \leq \kappa(x) L^{n+1} \mathbb{1}(x)$$

*for all  $x \in \mathbf{X}$  and all  $n \in \{1, \dots, m-1\}$ . Moreover, Assumption 6.1 (a) and (c) hold.*

By Lemma 6.2, it remains to show  $\rho(L) < 1$  to ensure Assumption 6.1 (b). Since  $\sup_{z \in \mathbf{Z}} \mathbb{E}_z \beta_0 \beta_1 \beta_2 \cdots \beta_{m-1} (\mathbb{E} R(\varepsilon'))^m < 1$  for some  $m \in \mathbb{N}$ , we have  $\|L^m \mathbb{1}\| < 1$  and then  $\rho(L) < 1$ . Therefore, Assumption 6.1 is satisfied, and then all conclusions of Theorem 6.1 hold.

## 7. EXTENSION: RISK-SENSITIVE PREFERENCES

We study the risk-sensitive preference models with state-action-dependent discounting in this section, where the agent is risk-averse in future utility and future consumption. Let an MDP satisfy the following regular conditions.

**Condition 7.1.**

- (i)  $(x, a) \mapsto r(x, a)$  is u.s.c.,
- (ii)  $\beta$  is bounded and strictly positive,
- (iii)  $\Gamma$  is nonempty, compact-valued, and continuous,
- (iv)  $(x, a) \mapsto (-\beta(x, a)/\theta) \ln \int \exp(-\theta h(y)) P(x, a, dy)$  is u.s.c. for any  $h \in cb\mathbf{X}$ ,

Let  $\theta > 0$  be the agents' risk-sensitive coefficient. For any feasible policy  $\sigma \in \Sigma$  and Borel measurable function  $v: \mathbf{X} \rightarrow \mathbb{R} \cup \{-\infty\}$ , let

$$T_\sigma v(x) := r(x, \sigma(x)) - \frac{\beta(x, \sigma(x))}{\theta} \ln \mathbb{E}_x^\sigma e^{-\theta v(X')} \quad (x \in \mathbf{X}). \quad (20)$$

The  $\sigma$ -value function is defined by

$$v_\sigma(x) = \limsup_{x \rightarrow \infty} T_\sigma^n \bar{r}(x) \quad (x \in \mathbf{X}).$$

The associated Bellman equation is

$$v(x) = \sup_{a \in \Gamma(x)} \left\{ r(x, a) - \frac{\beta(x, a)}{\theta} \ln \mathbb{E}_{x,a} e^{-\theta v(X')} \right\} \quad (x \in \mathbf{X}). \quad (21)$$

Suppose that the state process evolves following

$$x_{t+1} = f(x_t, a_t, \varepsilon_{t+1}), \quad (22)$$

where  $f$  is Borel measurable, and  $\{\varepsilon_t\}$  is an IID process taking values in  $\mathbb{R}^m$ . For each  $t$ , let  $\varepsilon_t = (\varepsilon_{1t}, \dots, \varepsilon_{mt})$ .

**Assumption 7.1.**

- (i)  $r: \mathbf{G} \rightarrow \mathbb{R}$  and  $\beta: \mathbf{G} \rightarrow \mathbb{R}$  are increasing in  $x$ ,
- (ii)  $r_\sigma$  and  $\beta_\sigma$  are increasing for any  $\sigma \in \Sigma$
- (iii)  $f(x, a, \varepsilon')$  is increasing in  $(x, \varepsilon')$ , and  $f(x, \sigma(x), \varepsilon')$  is increasing in  $(x, \varepsilon')$  for any  $\sigma \in \Sigma$ ,
- (iv)  $\Gamma(x_1) \subset \Gamma(x_2)$  if  $x_1 \leq x_2$ ,

(v)  $\varepsilon_{1t}, \dots, \varepsilon_{mt}$  are independent for each  $t$ .

Let  $g(x, a) := -\beta(x, a)/\theta \log \mathbb{E}_{x,a} \exp(-\theta v(X'))$  for all  $(x, a) \in \mathbf{G}$ . Let  $\mathcal{G}$  be the u.s.c. functions  $g$  on  $\mathbf{G}$  such that  $\|g\| < \infty$  and  $g$  is increasing in  $\mathbf{X}$ . Similar to (17), the transformed Bellman equation is

$$g(x, a) = -\frac{\beta(x, a)}{\theta} \ln \mathbb{E}_{x,a} \exp \left( -\theta \sup_{a' \in \Gamma(x')} \{r(X', a') + g(X', a')\} \right). \quad (23)$$

for  $(x, a) \in \mathbf{G}$  and  $g \in \mathcal{G}$ . Define the *risk-sensitive expected value Bellman operator*  $R$  by letting  $Rg(x, a)$  be the right-hand side of (23) for any  $g \in \mathcal{G}$  and  $(x, a) \in \mathbf{G}$ .

**Assumption 7.2.**  $\bar{r}$  is bounded above and  $\hat{r}$  is bounded below, where  $\bar{r}$  is defined by (16) and  $\hat{r}$  is defined by

$$\hat{r}(x, a) := -\frac{1}{\theta} \ln \mathbb{E}_{x,a} \exp(-\theta \bar{r}(X')) \quad ((x, a) \in \mathbf{G}).$$

Define  $\bar{\beta}(x) := \sup_{a \in \Gamma(x)} \beta(x, a)$  for all  $x \in \mathbf{X}$ . Suppose that there exists an  $\ell: \mathbf{X}^2 \rightarrow \mathbb{R}_+$  such that  $x \mapsto \ell(x, B)$  is a measurable function for all  $B \in \mathcal{B}(\mathbf{X})$ ,  $B \mapsto \ell(x, B)$  is a measure on  $\mathbf{X}$ , and  $L: mb\mathbf{X} \rightarrow mb\mathbf{X}$  is defined by

$$Lh(x) := \int \ell(x, dx') h(x')$$

for  $x \in \mathbf{X}$  and  $h \in mb\mathbf{X}$ .

**Assumption 7.3.**  $\rho(L) < 1$ , and  $\int_{\mathbf{X}} h(x') \bar{\beta}(x) P(x, a, dx') \leq Lh(x)$  for all  $(x, a) \in \mathbf{G}$  and  $h \in mb\mathbf{X}_+$ .

**Lemma 7.1.** *If Condition 7.1 and , then  $R\mathcal{G} \subset \mathcal{G}$ .*

*Proof.* Let Condition 7.1 and Assumption 7.2 hold. Let  $g \in \mathcal{G}$ . We first show that  $\|Rg\| < \infty$ . Since  $g \geq -\|g\|$ , we have

$$\begin{aligned} Rg(x, a) &\geq -\frac{\beta(x, a)}{\theta} \ln \mathbb{E}_{x,a} \exp \left( -\theta \sup_{a' \in \Gamma(x')} \{r(X', a') - \|g\|\} \right) \\ &= -\frac{\beta(x, a)}{\theta} \ln \mathbb{E}_{x,a} \exp(-\theta(\bar{r}(X') - \|g\|)) = \beta(x, a)(\hat{r}(x, a) - \|g\|). \end{aligned}$$

Hence, since  $\hat{r}$  is bounded below,  $Rg$  is also bounded below. Similarly, we can show that  $Rg(x, a) \leq \beta(x, a)(\hat{r}(x, a) + \|g\|)$  for all  $(x, a) \in \mathbf{G}$ . Since  $\beta$  is bounded and  $\bar{r}$  is

bounded above,  $\hat{r}$  is bounded above and then  $Rg$  is bounded above. Therefore, we have  $\|Rg\| < \infty$ .

Next, since  $g$  and  $r$  are u.s.c., and  $\Gamma$  is compact-valued and u.s.c., the map  $x \mapsto h(x) := \sup_{a \in \Gamma(x)} \{r(x, a) + g(x, a)\}$  is u.s.c. by the Maximum theorem. Now, note that if  $x_1 \leq x_2$  in  $\mathbf{X}$ , then  $x'_1 = f(x_1, a, \varepsilon') \leq f(x_2, a, \varepsilon') = x'_2$  and then  $\Gamma(x'_1) \subset \Gamma(x'_2)$ . Since  $r$  and  $g$  are increasing in  $x$ , we have

$$h(x'_1) = \sup_{a' \in \Gamma(x'_1)} \{r(x'_1, a') + g(x'_1, a')\} \leq \sup_{a' \in \Gamma(x'_2)} \{r(x'_2, a') + g(x'_2, a')\} = h(x'_2).$$

Therefore, we have  $-(1/\theta) \ln \mathbb{E}_{x_1, a} \exp(-\theta h(x'_1)) \leq -(1/\theta) \ln \mathbb{E}_{x_2, a} \exp(-\theta h(x'_2))$ . Since in addition  $\beta$  is increasing in  $x$ , we see that  $Rg$  is increasing in  $x$ . We conclude that  $R\mathcal{G} \subset \mathcal{G}$ .  $\square$

**Lemma 7.2.** *If Condition 7.1, Assumption 7.1, 7.2, and 7.3 hold, then  $R$  is eventually contracting on  $(\mathcal{G}, \|\cdot\|)$  and has a unique fixed point in  $\mathcal{G}$ .*

*Proof.* Let Condition 7.1, Assumption 7.1, 7.2, and 7.3 hold. We show that there is an  $n \in \mathbb{N}$  such that  $R^n$  satisfies Blackwell's condition. Let  $g \in \mathcal{G}$  and  $K \geq 0$ . Fix  $(x, a) \in \mathbf{G}$ . Observe that

$$\begin{aligned} R(g + K)(x, a) &= \frac{-\beta(x, a)}{\theta} \ln \mathbb{E}_{x, a} \exp \left( -\theta \sup_{a' \in \Gamma(X')} \{r(X', a') + g(X', a') + K\} \right) \\ &= Rg(x, a) + \beta(x, a)K \leq Rg(x, a) + KL\mathbb{1}(x). \end{aligned}$$

Define the function  $\varphi(t) = -1/\theta \ln \mathbb{E}_{x, a} \exp(-\theta t)$  for random variable  $t = t(X')$ . Since  $\varphi$  is monotonically increasing, iteration implies

$$\begin{aligned} R^2(g + K)(x, a) &= \beta(x, a) \varphi \left( \sup_{a' \in \Gamma(X')} \{r(X', a') + R(g + K)(X', a')\} \right) \\ &= \beta(x, a) \varphi \left( \sup_{a' \in \Gamma(X')} \{r(X', a') + Rg(X', a') + \beta(X', a')K\} \right) \\ &\leq \beta(x, a) \varphi \left( \sup_{a' \in \Gamma(X')} \{r(X', a') + Rg(X', a')\} + K \sup_{a' \in \Gamma(X')} \{\beta(X', a')\} \right). \end{aligned}$$

Let  $X = \sup_{a' \in \Gamma(X')} \{r(X', a') + Rg(X', a')\}$  and  $Y = K \sup_{a' \in \Gamma(X')} \{\beta(X', a')\}$ . Since  $Rg(x, a)$ ,  $r(x, a)$ ,  $\beta(x, a)$ , and  $\Gamma(x)$  are increasing in  $x$ , and  $x' = f(x, a, \varepsilon')$  is increasing

in  $x$  and  $\varepsilon'$  for independent  $\varepsilon'$ , we have  $\text{Cov}(e^{-\theta X}, e^{-\theta Y} | (x, a)) \geq 0$ . Therefore, since  $\mathbb{E}_{x,a} e^{-\theta X} e^{-\theta Y} \geq \mathbb{E}_{x,a} e^{-\theta X} \mathbb{E}_{x,a} e^{-\theta Y}$ , we have<sup>19</sup>

$$\begin{aligned} \varphi(X + Y) &= \frac{-1}{\theta} \ln \mathbb{E}_{x,a} e^{-\theta X - \theta Y} \\ &\leq \frac{-1}{\theta} \ln (\mathbb{E}_{x,a} e^{-\theta X} \mathbb{E}_{x,a} e^{-\theta Y}) = \varphi(X) + \varphi(Y). \end{aligned}$$

We then have

$$\begin{aligned} R^2(g + K)(x, a) &\leq \beta(x, a) \varphi \left( \sup_{a' \in \Gamma(X')} \{r(X', a') + Rg(X', a')\} \right) + \beta(x, a) \varphi \left( K \sup_{a' \in \Gamma(X')} \{\beta(X', a')\} \right) \\ &= R^2g(x, a) + \beta(x, a) \varphi \left( K \sup_{a' \in \Gamma(X')} \{\beta(X', a')\} \right). \end{aligned}$$

Now, since  $t \mapsto \exp(-\theta t)$  is convex, Jensen inequality implies

$$\varphi(Y) = \frac{-1}{\theta} \ln \mathbb{E}_{x,a} e^{-\theta Y} \leq \frac{-1}{\theta} \ln e^{-\theta \mathbb{E}_{x,a} Y} = \mathbb{E}_{x,a} Y.$$

Therefore, we obtain

$$\begin{aligned} R^2(g + K)(x, a) &\leq R^2g(x, a) + K \mathbb{E}_{x,a} \beta(x, a) \sup_{a' \in \Gamma(X')} \beta(X', a') \\ &\leq R^2g(x, a) + KL\bar{\beta}(x), \end{aligned}$$

where  $L$  and  $\bar{\beta}$  are defined in Assumption 7.3. With the same argument, the induction shows that  $R^n(g + K)(x, a) \leq R^n g(x, a) + KL^{n-1} \bar{\beta}(x)$  for all  $n \in \mathbb{N}$ . Since  $\rho(L) < 1$  implies that there is  $m \in \mathbb{N}$  such that  $\|L^{m-1} \bar{\beta}\| < 1$ , we have

$$R^m(g + K) \leq R^m g + K \|L^{m-1} \bar{\beta}\|.$$

Hence,  $R^m$  satisfies the Blackwell's condition and is a contraction map. Then, since  $\mathcal{G}$  is a Banach space,  $R$  admits a unique fixed point in  $\mathcal{G}$  by the generalized Contracting Mapping theorem.  $\square$

**Lemma 7.3.** *If Condition 7.1, Assumption 7.1, 7.2, and 7.3 hold, then for all constant  $c \in \mathbb{R}$  we have  $v_\sigma(x) = \limsup_{n \rightarrow \infty} T_\sigma^n \bar{r}(x) = \limsup_{n \rightarrow \infty} T_\sigma^n(\bar{r} + c)(x)$  for all  $x \in \mathbf{X}$  and  $\sigma \in \Sigma$ .*

<sup>19</sup>See also Lemma 3 of [Bauerle and Jařkiewicz \(2018\)](#).

*Proof.* Let Condition 7.1, Assumption 7.1, 7.2, and 7.3 hold. Fix  $\sigma \in \Sigma$  and constant  $K \geq 0$ . Since  $\bar{r} + K \geq \bar{r}$ , the monotonicity of  $T_\sigma$  implies  $\limsup_{n \rightarrow \infty} T_\sigma^n(\bar{r} + K) \geq \limsup_{n \rightarrow \infty} T_\sigma^n \bar{r}$ . Next, similar to the iteration in Lemma 7.2, we have, for all  $x \in \mathbf{X}$ ,

$$\begin{aligned} T_\sigma(\bar{r} + K)(x) &= r_\sigma(x) - \frac{\beta_\sigma(x)}{\theta} \ln \mathbb{E}_x^\sigma e^{-\theta(\bar{r}+K)(X')} \\ &= r_\sigma(x) - \frac{\beta_\sigma(x)}{\theta} \ln e^{-\theta K} \mathbb{E}_x^\sigma e^{-\theta \bar{r}(X')} \\ &= r_\sigma(x) - \frac{\beta_\sigma(x)}{\theta} \ln \mathbb{E}_x^\sigma e^{-\theta \bar{r}(X')} + \beta_\sigma(x)K = T_\sigma \bar{r}(x) + \beta_\sigma(x)K, \end{aligned}$$

where  $r_\sigma(x) = r(x, \sigma(x))$  and  $\beta_\sigma(x) = \beta(x, \sigma(x))$  for all  $x \in \mathbf{X}$ . Since  $r_\sigma$ ,  $\beta_\sigma$ ,  $x' = f(x, \sigma(x), \varepsilon')$  are increasing in  $x$ , we see that  $T_\sigma \bar{r}(x)$  is increasing in  $x$ . Then, it follows from the argument in Lemma 7.2 and Assumption 7.1 that  $\mathbb{E}_x^\sigma e^{-\theta T_\sigma \bar{r}(X')} e^{-\theta \beta_\sigma(X')K} \geq \mathbb{E}_x^\sigma e^{-\theta T_\sigma \bar{r}(X')} \mathbb{E}_x^\sigma e^{-\theta \beta_\sigma(X')K}$ , so iteration yields that for all  $x \in \mathbf{X}$

$$\begin{aligned} T_\sigma^2(\bar{r} + K)(x) &= r_\sigma(x) - \frac{\beta_\sigma(x)}{\theta} \ln \mathbb{E}_x^\sigma e^{-\theta T_\sigma(\bar{r}+K)(X')} \\ &= r_\sigma(x) - \frac{\beta_\sigma(x)}{\theta} \ln \mathbb{E}_x^\sigma e^{-\theta(T_\sigma \bar{r}(X') + \beta_\sigma(X')K)} \\ &= r_\sigma(x) - \frac{\beta_\sigma(x)}{\theta} \ln \mathbb{E}_x^\sigma e^{-\theta T_\sigma \bar{r}(X')} e^{-\theta \beta_\sigma(X')K} \\ &\leq r_\sigma(x) - \frac{\beta_\sigma(x)}{\theta} \ln \mathbb{E}_x^\sigma e^{-\theta T_\sigma \bar{r}(X')} \mathbb{E}_x^\sigma e^{-\theta \beta_\sigma(X')K} \\ &= T_\sigma^2 \bar{r}(x) - \frac{\beta_\sigma(x)}{\theta} \ln \mathbb{E}_x^\sigma e^{-\theta \beta_\sigma(X')K} \\ &\leq T_\sigma^2 \bar{r}(x) + K \mathbb{E}_x^\sigma \beta_\sigma(x) \beta_\sigma(X') \\ &\leq T_\sigma^2 \bar{r}(x) + KL \bar{\beta}(x) \end{aligned}$$

where the second inequality follows from Jensen inequality as iteration in Lemma 7.2, and the last inequality follows from Assumption 7.3. Then, using the same induction, we have

$$T_\sigma^n(\bar{r} + K)(x) \leq T_\sigma^n \bar{r}(x) + KL^{n-1} \bar{\beta}(x).$$

Hence, since  $\rho(L) < 1$  by Assumption 7.3, there exists  $m \in \mathbb{N}$  such that  $\|L^m \bar{\beta}\| < 1$ . Then, we have  $\limsup_{n \rightarrow \infty} L^n \bar{\beta}(x) \leq \limsup_{n \rightarrow \infty} \|L^n \bar{\beta}\| \rightarrow 0$  for all  $x \in \mathbf{X}$ . Therefore, we have  $\limsup_{n \rightarrow \infty} T_\sigma^n(\bar{r} + K)(x) \leq \limsup_{n \rightarrow \infty} T_\sigma^n \bar{r}(x)$  for all  $x \in \mathbf{X}$ . Then, we have  $\limsup_{n \rightarrow \infty} T_\sigma^n(\bar{r} + K)(x) = \limsup_{n \rightarrow \infty} T_\sigma^n \bar{r}(x)$  for all  $x \in \mathbf{X}$  when  $K \geq 0$ . Similarly, we can show the statement when  $K < 0$ , where the above inequalities are reversed.  $\square$



If Assumption 7.1, 7.2, and 7.3 are satisfied, the optimality results follow from the following theorem.

**Theorem 7.1.** *If Condition 7.1, Assumption 7.1, 7.2, and 7.3 hold, then the following statements are true.*

- (a)  $R\mathcal{G} \subset \mathcal{G}$  and  $R$  is eventually contracting on  $(\mathcal{G}, \|\cdot\|)$ ,
- (b)  $R$  admits a unique fixed point  $g^*$  in  $\mathcal{G}$ ,
- (c)  $v^*$  is well defined and

$$v^*(x) = \sup_{a \in \Gamma(x)} \{r(x, a) + g^*(x, a)\} \quad g^*(x, a) = -\frac{\beta(x, a)}{\theta} \log \mathbb{E}_{x, a} e^{-\theta v^*(X')},$$

- (d) AFI converges to  $g^*$ ,
- (e) at least one optimal policy exists, and
- (f) a feasible policy is optimal if and only if it is  $g^*$ -greedy.

## APPENDIX A. APPENDIX

We quote some useful theorems in this section, including the generalized Contraction Mapping Theorem, the Maximum Theorem, and Gelfand's formula.

**Theorem A.1** (generalized Contraction Mapping Theorem). *If  $F$  is a self-map in a metric space  $X$  and  $F^m$  is contractive for some  $m \in \mathbb{N}$ , then  $F$  has a unique fixed point, which is the limit of every sequence  $\{F^k x\}$  for arbitrary  $x \in X$ .*

See theorem 4 of [Cheney \(2001\)](#) for the proof of the following contraction mapping theorem. Let  $X$  and  $Y$  be topological spaces. In the following theorem, given a correspondence  $\varphi: X \rightarrow Y$ , denote  $\text{Gr}(\varphi) := \{(x, y): x \in X, y \in \varphi(x)\}$ .

**Theorem A.2** (Berge Maximum Theorem). *Let  $\varphi: X \rightarrow Y$  be a continuous correspondence between topological spaces with nonempty compact values. Suppose  $f: \text{Gr}(\varphi) \rightarrow \mathbb{R}$  is continuous. Define the value function  $m: X \rightarrow \mathbb{R}$  by*

$$m(x) := \max_{y \in \varphi(x)} f(x, y),$$

*and the correspondence  $\mu: X \rightarrow Y$  of maximizers by*

$$\mu(x) := \{y \in \varphi(x): f(x, y) = m(x)\}.$$

*Then:*

- (a) *The value function  $m$  is continuous.*
- (b) *The  $\arg \max$  correspondence  $\mu$  has nonempty compact values.*
- (c) *If either  $f$  has a continuous extension to all  $X \times Y$  or  $Y$  is Hausdorff, then the  $\arg \max$  correspondence  $\mu$  is upper semicontinuous.*

See Chapter 17 of [Aliprantis and Border \(2006\)](#) and [Hernández-Lerma and Lasserre \(2012a\)](#) for the detail of the Maximum theorem.

Let  $E$  be a Banach space. A closed convex set  $K \subset E$  is called a *cone* if  $x \in K$  implies  $tx \in K$  for  $t \geq 0$ , and  $x, -x \in K$  implies  $x = \theta$ , where  $\theta$  denotes the zero element of  $E$ . A cone  $K$  is solid if it has an interior point; it is normal if  $\theta \preceq x \preceq y$  implies  $\|x\| \leq M\|y\|$  for  $x, y \in K$ , where  $\preceq$  denotes partial order on  $E$ , and  $M$  is a constant.<sup>20</sup> The cone of non-negative functions in  $mb\mathbf{X}$  or  $cb\mathbf{X}$  is both solid and normal.

**Theorem A.3** (Gelfand's Formula). *Let  $A$  be a positive operator in a space  $E$  with a cone  $K$ . If the cone  $K$  is solid and normal, and  $f$  is an interior element of  $K$ , then*

$$\rho(A) = \lim_{n \rightarrow \infty} \|A^n f\|^{1/n}.$$

See Chapter 9 of [Krasnosel'skii et al. \(2012\)](#) for the proof of Gelfand's formula.

### A.1. Proofs in Section 2.

*Proof of Lemma 2.1.* Let Condition 2.1 hold. Clearly, since  $\sigma \in \Sigma$  is measurable, Condition 2.1 implies that  $T_\sigma$  is a self-map on  $mb\mathbf{X}$ . We next show that  $T$  is a self-map on  $cb\mathbf{X}$ . Fix  $v \in cb\mathbf{X}$ . Since  $(x, a) \mapsto B(x, a, v)$  is bounded and continuous on  $\mathbf{G}$ , and  $\Gamma$  is continuous and compact-valued, it follows from the (measurable) Maximum theorem that  $x \mapsto \sup_{a \in \Gamma(x)} B(x, a, v) = Tv(x)$  is continuous and the correspondence  $\tau: \mathbf{X} \rightarrow \mathbf{A}$  defined by

$$\tau(x) = \arg \max_{a \in \Gamma(x)} B(x, a, v)$$

is nonempty, u.s.c, and compact-valued and admits a Borel measurable selector  $\sigma$  satisfying  $\sigma(x) \in \tau(x)$  for all  $x \in \mathbf{X}$ . Then,  $\sigma$  is a Borel measurable  $v$ -greedy policy. Since  $Tv(x) = B(x, a, v)$  for  $a \in \tau(x)$ ,  $Tv$  is bounded. Therefore,  $Tv$  is a self-map on  $cb\mathbf{X}$ .  $\square$

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<sup>20</sup>The definition of normal cone is equivalent to that there exists  $\delta > 0$  such that  $\|x + y\| \geq \delta$  for  $x, y \in K$  and  $\|x\| = \|y\| = 1$ .

*Proof of Lemma 2.2.* Let  $\mathcal{R}$  be regular. Fix  $\sigma \in \Sigma$  and let the linear operator  $L_\sigma$  be defined by (4). It follows from Theorem 1.5.5 of Bühler and Salamon (2018) that  $\rho(L_\sigma) := \lim_{n \rightarrow \infty} \|L_\sigma^n \mathbb{1}\|^{1/n}$  always exists and is bounded above by  $\|L_\sigma\|$ . Applying Gelfand's formula, since (i)  $L_\sigma$  is a positive linear operator on  $mb\mathbf{X}$ , (ii) the positive cone in  $mb\mathbf{X}$  is solid and normal under the pointwise partial order, and (iii)  $\mathbb{1}$  lies interior to the positive cone in  $mb\mathbf{X}$ , we obtain

$$\rho(L_\sigma) = \lim_{n \rightarrow \infty} \|L_\sigma^n \mathbb{1}\|^{1/n} = \lim_{n \rightarrow \infty} \left\{ \sup_x |L_\sigma^n \mathbb{1}(x)| \right\}^{1/n} = \lim_{n \rightarrow \infty} (d_n^\sigma)^{1/n}. \quad (24)$$

To show (a), suppose  $\rho(L_\sigma) < 1$ . Then, (24) implies that there is an  $n_\sigma \in \mathbb{N}$  such that  $d_{n_\sigma}^\sigma < 1$ . Conversely, suppose that  $d_{n_\sigma}^\sigma < 1$  for some  $n_\sigma \in \mathbb{N}$ . Denote  $\ell_\sigma(x, dx') = k(x, \sigma(x), dx')$ . Since any  $n \in \mathbb{N}$  can be written uniquely as  $n = kn_\sigma + i$  for some  $k, i \in \mathbb{N}_0$  with  $i < n_\sigma$ , we have that, for sufficiently large  $n$ ,

$$\begin{aligned} d_n^\sigma &= \sup_x L_\sigma^n \mathbb{1}(x) = \sup_x L_\sigma^{kn_\sigma} L_\sigma^{n-kn_\sigma} \mathbb{1}(x) \\ &= \sup_x \int_{\mathbf{X}} \ell_\sigma^{kn_\sigma}(x, dx') \int_{\mathbf{X}} \mathbb{1}(x'') \ell_\sigma^{n-kn_\sigma}(x', dx'') \\ &\leq \sup_x \int_{\mathbf{X}} \ell_\sigma^{kn_\sigma}(x, dx') \left( \sup_{x'} \int_{\mathbf{X}} \ell_\sigma^{n-kn_\sigma}(x', dx'') \right) \\ &= d_{kn_\sigma}^\sigma d_{n-kn_\sigma}^\sigma \end{aligned} \quad (25)$$

where the second inequality follows from the same argument for the first inequality. Then, we have

$$(d_n^\sigma)^{1/n} = (d_{kn_\sigma}^\sigma d_{n-kn_\sigma}^\sigma)^{1/n} \leq (d_{n_\sigma}^\sigma)^{k/n} (d_i^\sigma)^{1/n} \leq (d_{n_\sigma}^\sigma)^{k/n} (M)^{1/n}.$$

Since  $k/n \rightarrow 1/n_\sigma$  as  $n \rightarrow \infty$ , the right-hand side converges to  $(d_{n_\sigma}^\sigma)^{1/n_\sigma} < 1$  as  $n \rightarrow \infty$ . Hence,  $\rho(L_\sigma) < 1$ . Since  $\sigma$  is arbitrarily chosen, the statements hold for all  $\sigma \in \Sigma$ . Part (c) follows from Gelfand's formula with the fact  $L_\sigma^n \mathbb{1} \leq \hat{L}^n \mathbb{1}$  for  $n \in \mathbb{N}$  by iteration.  $\square$

**Lemma A.1.** *If  $\mathcal{R}$  is regular, and  $\rho(L_\sigma) < 1$  for all  $\sigma \in \Sigma$ , then  $T_\sigma$  is eventually contracting, globally stable on  $mb\mathbf{X}$ , and has a unique fixed point  $v_\sigma \in mb\mathbf{X}$  for all  $\sigma \in \Sigma$ . Moreover, if  $\sigma \in \Sigma$  is continuous, then  $v_\sigma \in cb\mathbf{X}$ .*

*Proof of Lemma A.1.* Suppose that  $\mathcal{R}$  is regular and  $\rho(L_\sigma) < 1$  for all  $\sigma \in \Sigma$ . Fix  $\sigma \in \Sigma$ . Let  $L_\sigma$  be the operator defined by (4). Let  $v, w \in mb\mathbf{X}$ . Then, we have

$$|T_\sigma v(x) - T_\sigma w(x)| \leq L_\sigma |v - w|(x) \quad (x \in \mathbf{X}).$$

Since  $L_\sigma$  is order-preserving, iteration implies that  $|T_\sigma^n v - T_\sigma^n w| \leq L_\sigma^n |v - w|$  for all  $n \in \mathbb{N}$ . Since  $\rho(L_\sigma) < 1$ , Lemma 2.2 implies that there is an  $n_\sigma \in \mathbb{N}$  such that  $d_{n_\sigma}^\sigma < 1$  and iteration implies

$$L_\sigma^{n_\sigma} h(x) \leq d_{n_\sigma}^\sigma \|h\| \quad (h \in mb\mathbf{X}.)$$

Therefore, we have

$$|T_\sigma^{n_\sigma} v - T_\sigma^{n_\sigma} w| \leq L_\sigma^{n_\sigma} |v - w| \leq d_{n_\sigma}^\sigma \|v - w\|.$$

Taking supremum over  $\mathbf{X}$  on the left, we have  $\|T_\sigma^{n_\sigma} v - T_\sigma^{n_\sigma} w\| \leq d_{n_\sigma}^\sigma \|v - w\|$ . We conclude that  $T_\sigma^{n_\sigma}$  is a contracting map with modulus  $d_{n_\sigma}^\sigma$ . Finally, the generalized Contraction Mapping theorem shows that  $T_\sigma$  is globally stable and has a unique fixed point  $v_\sigma \in mb\mathbf{X}$ . If  $\sigma \in \Sigma$  is continuous, then  $T_\sigma$  is a self-map on  $cb\mathbf{X}$ . Then,  $T_\sigma$  is globally stable on both  $cb\mathbf{X}$  and  $mb\mathbf{X}$ . Since  $cb\mathbf{X} \subset mb\mathbf{X}$  and  $v_\sigma$  is unique in both  $mb\mathbf{X}$  and  $cb\mathbf{X}$ , we have  $v_\sigma \in cb\mathbf{X}$ .  $\square$

**Lemma A.2.** *If  $\mathcal{R}$  is regular and Assumption 2.2 holds, then  $\rho(L_\sigma) < 1$  and  $T_\sigma$  has a unique fixed point and is eventually contracting and globally stable on  $mb\mathbf{X}$  for all  $\sigma \in \Sigma$ .*

*Proof of Lemma A.2.* Suppose that  $\mathcal{R}$  is regular and Assumption 2.2 holds. Fix  $\sigma \in \Sigma$ . By definition, we have  $L_\sigma \mathbb{1} \leq \hat{L} \mathbb{1}$ , which implies  $L_\sigma^2 \mathbb{1} \leq \hat{L} L_\sigma \mathbb{1} \leq \hat{L}^2 \mathbb{1}$ . Iteration yields  $L_\sigma^n \mathbb{1} \leq \hat{L}^n \mathbb{1}$  for all  $n \in \mathbb{N}$ . Since  $\rho(\hat{L}) < 1$ , there is an  $m \in \mathbb{N}$  such that  $\|L_\sigma^m \mathbb{1}\| \leq \|\hat{L}^m \mathbb{1}\| < 1$ . This implies  $\rho(L_\sigma) < 1$ . Fix  $v, v' \in mb\mathbf{X}$ . Since  $T_\sigma v(x) - T_\sigma v'(x) \leq L_\sigma(v - v')(x)$  for  $x \in \mathbf{X}$ , we have  $|T_\sigma(x)v - T_\sigma v'(x)| \leq L_\sigma |v - v'| (x)$  for all  $x \in \mathbf{X}$ . Iteration implies  $|T_\sigma^m v - T_\sigma^m v'| \leq L_\sigma^m |v - v'| \leq \|v - v'\| L_\sigma^m \mathbb{1}$ . Taking supremum, we have  $\|T_\sigma^m v - T_\sigma^m v'\| \leq \|v - v'\| \|L_\sigma^m \mathbb{1}\|$ . Then,  $T_\sigma^m$  is contracting, so the generalized Contraction Theorem concludes the statement.  $\square$

**Lemma A.3.** *If  $\mathcal{R}$  is regular and Assumption 2.2 holds, then  $T$  has a unique fixed point and is eventually contracting and globally stable on  $cb\mathbf{X}$ .*

*Proof of Lemma A.3.* Suppose that  $\mathcal{R}$  is regular and Assumption 2.2 holds. Then,  $T$  is a self-map on  $cb\mathbf{X}$ . Fix  $v, v' \in cb\mathbf{X}$ . We have

$$\begin{aligned} |Tv(x) - Tv'(x)| &\leq \sup_{\sigma} |T_{\sigma}v(x) - T_{\sigma}v'(x)| \\ &\leq \sup_{\sigma} |L_{\sigma}(v - v')(x)| \\ &\leq \sup_{\sigma} L_{\sigma}|v - v'| (x) = \hat{L}|v - v'| (x) \end{aligned}$$

for all  $x \in \mathbf{X}$ . Then, iteration yields

$$|T^n v(x) - T^n v'(x)| \leq \hat{L}^n |v - v'| (x)$$

for all  $x \in \mathbf{X}$ , which implies  $\|T^n v - T^n v'\| \leq \|\hat{L}^n \mathbb{1}\| \|v - v'\|$ . Since  $\rho(\hat{L}) < 1$ , there exist  $n \in \mathbb{N}$  such that  $\|\hat{L}^n \mathbb{1}\| < 1$ . Therefore,  $\hat{L}^n$  is contracting, and then the generalized Contraction Mapping shows that  $\hat{L}$  has a unique fixed point and globally stable.  $\square$

**Lemma A.4.** *If  $\mathcal{R}$  is regular, and Assumption 2.2 holds, then  $T$  is contracting on  $(cb\mathbf{X}, \|\cdot\|_w)$ , where  $w := \sum_{n=0}^{\infty} \hat{L}^n \mathbb{1}$ .*

*Proof of Lemma A.4.* Suppose that  $\mathcal{R}$  is regular, and Assumption 2.2 holds. Then,  $\rho(\hat{L}) < 1$ . Let  $\{f_n\}_{n \geq 0}$  be a monotonically increasing sequence in  $mb\mathbf{X}$  such that  $v_n \uparrow v \in mb\mathbf{X}$  pointwise. Then, since  $\hat{L}$  and  $L_{\sigma}$  preserve orders, the Dominated Convergence Theorem implies that

$$\begin{aligned} \lim_{n \rightarrow \infty} \hat{L}v_n(x) &= \sup_n \sup_{\sigma \in \Sigma} L_{\sigma}v_n(x) \\ &= \sup_{\sigma \in \Sigma} \sup_n \int_{\mathbf{X}} v_n(x) k(x, \sigma(x), dx') \\ &= \sup_{\sigma \in \Sigma} \int_{\mathbf{X}} \sup_n v_n(x) k(x, \sigma(x), dx') \\ &= \sup_n \int_{\mathbf{X}} v(x) k(x, \sigma(x), dx') = \hat{L}v(x) \end{aligned}$$

for  $x \in \mathbf{X}$ . Therefore, Assumption 8 of Bloise et al. (2024) holds. It then follows from Claim 6 of Bloise et al. (2024) that there are  $\lambda \in (0, 1)$  and everywhere positive  $w \in mb\mathbf{X}$  such that  $\hat{L}w \leq \lambda w$ . Note that we can construct  $w$  by  $w = \sum_{n=0}^{\infty} \hat{L}^n \mathbb{1}$ .<sup>21</sup>

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<sup>21</sup>See the proof of Claim 6, Bloise et al. (2024).

Fix  $v, v' \in cb\mathbf{X}$ . Then, we have

$$\begin{aligned}
|Tv(x) - Tv'(x)| &= \left| \sup_{\sigma} T_{\sigma}v(x) - \sup_{\sigma} T_{\sigma}v'(x) \right| \\
&\leq \sup_{\sigma} |T_{\sigma}v(x) - T_{\sigma}v'(x)| \\
&\leq \sup_{\sigma} L_{\sigma}(|v - v'|)(x) = \hat{L}(|v - v'|)(x) \\
&\leq \hat{L}(\|v - v'\|_w w)(x) = \|v - v'\|_w \hat{L}(w)(x) \\
&\leq \|v - v'\|_w \lambda w(x)
\end{aligned}$$

for  $x \in \mathbf{X}$ . Dividing  $w$  and taking supremum, it implies that  $\|Tv - Tv'\|_w \leq \lambda \|v - v'\|_w$ .  $\square$

**Lemma A.5.** *If Assumption 2.1 holds, then there exists  $w \in \mathbb{R}_+^{\mathbf{X}}$  such that  $T$  is contractive on  $(\mathbb{R}^{\mathbf{X}}, \|\cdot\|_w)$  and has a unique fixed point in  $\mathbb{R}^{\mathbf{X}}$ .*

*Proof of Lemma A.5.* Let Assumption 2.1 hold. For  $s \in \Sigma$ , since  $\rho(L_{\sigma}) < 1$ , there exist  $\lambda_{\sigma} < 1$  and everywhere positive  $w_{\sigma} \in \mathbb{R}_+^{\mathbf{X}}$  such that  $L_{\sigma}w_{\sigma} \leq \rho(L_{\sigma})w_{\sigma}$ . Define  $T': \mathbb{R}^{\mathbf{X}} \rightarrow \mathbb{R}^{\mathbf{X}}$  by  $T'_{\sigma}v(x) = -B(x, \sigma(x), v)$  for all  $x \in \mathbf{X}$ ,  $v \in \mathbb{R}^{\mathbf{X}}$ , and  $\sigma \in \Sigma$ . Define  $T': \mathbb{R}^{\mathbf{X}} \rightarrow \mathbb{R}^{\mathbf{X}}$  by  $T'v = \min_{\sigma}(T'_{\sigma}v)$  for all  $v \in \mathbb{R}^{\mathbf{X}}$ . Then, we have  $T'v = -Tv$  for all  $v \in \mathbb{R}^{\mathbf{X}}$ . Fix  $\sigma \in \Sigma$  and  $v, v' \in \mathbb{R}^{\mathbf{X}}$ . Let  $w(x) = \min_{\sigma} w_{\sigma}(x)$  for all  $x \in \mathbf{X}$  and  $\lambda = \max_{\sigma} \lambda_{\sigma}$ . Since  $\Sigma$  is finite, we have  $\lambda < 1$ , and the finiteness of  $\mathbf{X}$  implies that  $w$  is everywhere positive. We observe that

$$\begin{aligned}
T'_{\sigma}v(x) - T'_{\sigma}v'(x) &= -B(x, \sigma(x), v) + B(x, \sigma(x), v') \leq L_{\sigma}(-v + v')(x) \\
&\leq L_{\sigma}(\|v - v'\|_w w)(x) \leq \|v - v'\|_w L_{\sigma}w_{\sigma}(x) \\
&\leq \|v - v'\|_w \lambda_{\sigma} w_{\sigma}(x) \leq \|v - v'\|_w \lambda w_{\sigma}(x)
\end{aligned}$$

for all  $x \in \mathbf{X}$ . It implies that

$$T'v(x) = \min_{\sigma} T'_{\sigma}v(x) \leq T'_{\sigma}v(x) \leq T'_{\sigma}v'(x) + \|v - v'\|_w \lambda w_{\sigma}(x)$$

for all  $x \in \mathbf{X}$ . Taking minimum on RHS over  $\Sigma$ , we have

$$T'v(x) \leq T'v'(x) + \|v - v'\|_w \lambda w(x)$$

for all  $x \in \mathbf{X}$ . Also, interchanging the role of  $v$  and  $v'$  yields

$$T'v'(x) \leq T'v(x) + \|v - v'\|_w \lambda w(x).$$

for all  $x \in \mathbf{X}$ . Hence, we have  $|T'v(x) - T'v'(x)| \leq \|v - v'\|_w \lambda w(x)$  for  $x \in \mathbf{X}$ . Dividing  $w(x)$  and taking supremum, we have  $\|T'v - T'v'\|_w \leq \lambda \|v - v'\|_w$ , which implies

$$\|Tv - Tv'\|_w = \| -T'v + T'v'\|_w \leq \lambda \|v - v'\|_w$$

for all  $v, v' \in \mathbb{R}^{\mathbf{X}}$ . Therefore,  $T$  is contractive on  $(\mathbb{R}^{\mathbf{X}}, \|\cdot\|_w)$  and has a unique fixed point in  $\mathbb{R}^{\mathbf{X}}$ .  $\square$

**Proposition A.1.** *If Assumption 2.1 holds, then HPI converges to  $\sigma^*$  in finitely many steps, OPI converges to  $\sigma^*$ , VTI converges to  $v^*$ ,  $v^*$  is the unique solution to Bellman equation, the Bellman's principle of optimality holds, and at least one optimal policy exists.*

*Proof of Proposition A.1.* Suppose that Assumption 2.1 holds. Then,  $T_\sigma$  is globally stable for all  $\sigma \in \Sigma$ . Since  $\Sigma$  is finite, then it follows from Chapter 8 or Chapter 9 of Sargent and Stachurski (2023) that HPI converges to  $\sigma^*$  in finitely many steps, and the other statements hold. Alternately, since Lemma A.5 implies the global stability of  $T$ , the statements is shown by Lemma A.7, A.9, A.10, and A.12.  $\square$

**Lemma A.6.** *If  $\mathcal{R}$  is regular and Assumption 2.3 holds, then  $T$  is eventually contracting and globally stable on  $cb\mathbf{X}$  and has a unique fixed point  $\bar{v} \in cb\mathbf{X}$ , and  $T_\sigma$  is eventually contracting and globally stable on  $mb\mathbf{X}$ , and have unique fixed points  $v_\sigma \in mb\mathbf{X}$  for all  $\sigma \in \Sigma$ .*

*Proof of Lemma A.6.* Let  $\mathcal{R}$  be regular and Assumption 2.3 hold. Lemma 2.1 implies that  $T$  is a self-map on  $cb\mathbf{X}$ . Fix  $v, w \in cb\mathbf{X}$ . Then, we have

$$\begin{aligned} |Tv(x) - Tw(x)| &= \left| \sup_{\sigma} T_{\sigma}v(x) - \sup_{\sigma} T_{\sigma}w(x) \right| \\ &\leq \sup_{\sigma} |T_{\sigma}v(x) - T_{\sigma}w(x)| \\ &\leq \sup_{\sigma} |B(x, \sigma(x), v) - B(x, \sigma(x), w)| \\ &\leq L|v - w|(x) \quad (x \in \mathbf{X}). \end{aligned}$$

Hence, we obtain  $|Tv - Tw| \leq L|v - w|$ . Since  $L$  preserves order, iteration gives

$$|T^n v - T^n w| \leq L^n |v - w| \leq L^n \mathbb{1} \|v - w\| \leq \|L^n \mathbb{1}\| \|v - w\|$$

for  $n \in \mathbb{N}$ . Taking the supremum on the left, we have  $\|T^n v - T^n w\| \leq \|L^n \mathbb{1}\| \|v - w\|$  for all  $n \in \mathbb{N}$ . Since  $L$  is a positive linear operator, we have  $\rho(L) = \lim_{n \rightarrow \infty} \|L^n \mathbb{1}\|^{1/n}$  by

Theorem 9.1 of [Krasnosel'skii et al. \(2012\)](#) as the proof of Lemma 2.2. Since  $\rho(L) < 1$ , there exists an  $m \in \mathbb{N}$  such that  $\|L^m \mathbb{1}\| < 1$ . We conclude that  $\|T^m v - T^m w\| \leq \|L^m \mathbb{1}\| \|v - w\|$  for all  $v, w \in cb\mathbf{X}$ . Therefore,  $T^m$  is contracting, so the generalized Contracting Mapping theorem shows that  $T$  is globally stable on  $\mathbf{X}$  and has a unique fixed point  $\bar{v} \in cb\mathbf{X}$ . The proof for  $T_\sigma$  is similar.  $\square$

**Lemma A.7.** *If  $\mathcal{R}$  is regular,  $T_\sigma$  is globally stable on  $mb\mathbf{X}$  for all  $\sigma \in \Sigma$ , and  $T$  is globally stable on  $cb\mathbf{X}$ , then the following statements are true.*

- (a) *If  $\bar{v}$  is the unique fixed point of  $T$  on  $cb\mathbf{X}$ , then  $v^* = \bar{v}$  and there exists a  $\sigma^* \in \Sigma$  such that  $v^* = v_{\sigma^*}$ .*
- (b) *If  $T$  has a non-continuous fixed point  $\bar{v} \in mb\mathbf{X}$ , then  $v^* < \bar{v}$  and  $v_\sigma < \bar{v}$  for all  $\sigma \in \Sigma$ .*

*Proof of Lemma A.7.* Let  $\mathcal{R}$  be regular and suppose that  $T_\sigma$  is globally stable on  $mb\mathbf{X}$  for all  $\sigma \in \Sigma$ , and  $T$  is globally stable on  $cb\mathbf{X}$ . We first show (a). Fix  $\sigma \in \Sigma$  and let  $\bar{v}$  be the unique fixed point of  $T$  on  $cb\mathbf{X}$ . Since  $\bar{v}$  is the fixed point of  $T$  and  $T\bar{v} = \sup_{\sigma \in \Sigma} T_\sigma \bar{v}$ , we obtain  $\bar{v} = T\bar{v} \geq T_\sigma \bar{v}$ . Since  $T_\sigma$  is order-preserving, iteration gives  $\bar{v} \geq T_\sigma \bar{v} \geq \dots \geq T_\sigma^n \bar{v}$  for  $n \in \mathbb{N}$ . In addition, since  $T_\sigma$  is globally stable, we have  $\bar{v} \geq v_\sigma$  as  $n \rightarrow \infty$ , which implies  $\bar{v} \geq v^*$  by taking supremum over  $\Sigma$ .

Conversely, Lemma 2.1 implies that there exists a  $\sigma^* \in \Sigma$  such that  $T_{\sigma^*} \bar{v} = T\bar{v}$ . Then, we have  $T_{\sigma^*} \bar{v} = \bar{v}$ , whence  $\bar{v} = v_{\sigma^*}$  by the uniqueness of fixed point of  $T_{\sigma^*}$ . By the definition of  $v^*$ , we obtain  $v^* \geq v_{\sigma^*} = \bar{v}$ . We conclude that  $v^* = \bar{v}$  and an optimal policy  $\sigma^*$  exists.

To show (b), assume that there is a non-continuous  $\bar{v} \in mb\mathbf{X}$  such that  $T\bar{v} = \bar{v}$ . Toward contradiction, suppose that there is  $\sigma \in \Sigma$  such that  $v_\sigma \geq \bar{v}$ . Then, we have  $v^* \geq v_\sigma \geq \bar{v}$ . Moreover, since  $\bar{v}$  is a fixed point of  $T$ , we have  $\bar{v} = T\bar{v} \geq T_\sigma \bar{v}$  for any  $\sigma \in \Sigma$ . Iteration implies  $\bar{v} \geq T_\sigma^n \bar{v}$ , so  $\bar{v} \geq v_\sigma$  as  $n \rightarrow \infty$  by the global stability of  $T_\sigma$ . It implies that  $\bar{v} \geq v^*$ . Then, we obtain  $\bar{v} = v^*$ , which is continuous by (a), contradicting with non-continuity. Therefore, we must have  $v_\sigma < \bar{v}$  for any  $\sigma \in \Sigma$ . The same argument also implies  $v^* < \bar{v}$ .  $\square$

**Lemma A.8.** *If Condition 2.2 holds, then the following statements are true.*

- (a)  *$T$  and  $T_\sigma$  are self-maps on  $cb\mathbf{X}$  for all  $\sigma \in \Sigma_C$ , and*



(b) for any  $v \in cb\mathbf{X}$  there exists a continuous  $v$ -greedy policy.

*Proof of Lemma A.8.* Let  $\mathcal{R}$  be regular and Assumption 2.2 holds. Then, if  $\sigma \in \Sigma$  is continuous and  $v \in cb\mathbf{X}$ , then  $x \mapsto T_\sigma(x) = B(x, \sigma(x), v)$  is continuous. Hence,  $T_\sigma$  is a self-map on  $cb\mathbf{X}$ . Since  $\Gamma$  is convex-valued and compact-valued, and  $a \mapsto B(x, a, v)$  is strictly quasi-concave for any  $x \in \mathbf{X}$  and  $v \in cb\mathbf{X}$ , it follows from the Maximum theorem that the  $Tv$  is continuous and then  $T$  is a self-map on  $cb\mathbf{X}$ . Moreover, for any  $v \in cb\mathbf{X}$ , the maximizer correspondence  $x \mapsto \arg \max_{a \in \Gamma(x)} B(x, a, v)$  is single-valued and continuous, whence a continuous  $v$ -greedy policy exists.  $\square$

**Lemma A.9** (Bellman's Principle of Optimality). *Suppose that  $\mathcal{R}$  is regular and  $T_\sigma$  is globally stable for all  $\sigma \in \Sigma$ . If  $v^*$  is a fixed point to  $T$ , then Bellman's principle of optimality holds.*

*Proof of Lemma A.9.* Let all the stated assumptions hold. We want to show:  $\sigma \in \Sigma$  is optimal (i.e.,  $v_\sigma = v^*$ ) if and only if  $\sigma$  is  $v^*$ -greedy. Suppose that  $\sigma \in \Sigma$  is  $v^*$ -greedy:  $T_\sigma v^* = Tv^*$ . Since  $Tv^* = v^*$ , we obtain  $T_\sigma v^* = v^*$  so that  $v_\sigma = v^*$ . Conversely, suppose that  $\sigma \in \Sigma$  is optimal:  $v_\sigma = v^*$ . Since  $T_\sigma$  has a unique fixed point  $v_\sigma$ , we obtain  $v^* = T_\sigma v^*$ . Since  $Tv^* = v^*$ , we have  $T_\sigma v^* = v^* = Tv^*$ , so that  $\sigma$  is  $v^*$ -greedy.  $\square$

**Lemma A.10** (HPI). *Suppose that Condition 2.2 holds,  $T_\sigma$  is globally stable on  $cb\mathbf{X}$  with fixed point  $v_\sigma \in cb\mathbf{X}$  for all  $\sigma \in \Sigma_C$ , and  $T$  is globally stable on  $cb\mathbf{X}$ . If  $v^* \in cb\mathbf{X}$  is the unique fixed point of  $T$  on  $cb\mathbf{X}$ , then  $\{H^n v_{\sigma_0}\}_{n \geq 0}$  converges to  $v^*$  for any  $\sigma_0 \in \Sigma_C$ , and  $v^*$  is the unique fixed point of  $H$ .*

*Proof of Lemma A.10.* Let all the stated assumptions hold. Let  $\{\sigma_k\}_{k \geq 0} \subset \Sigma_C$  be such that  $\sigma_0 \in \Sigma_C$  and  $v_{\sigma_k} = Hv_{\sigma_{k-1}}$  for all  $k \in \mathbb{N}$ ; that is,  $\sigma_k \in \Sigma_C$  satisfies  $T_{\sigma_k} v_{\sigma_{k-1}} = Tv_{\sigma_{k-1}}$ . Note that for any  $k \geq 0$ , the continuity of  $\sigma_k$  implies that  $T_{\sigma_k}$  is a self-map on  $cb\mathbf{X}$ ,  $v_{\sigma_k}$  is continuous, and then a continuous  $v_{\sigma_k}$ -greedy policy  $\sigma_{k+1}$  exists by Lemma A.8. By definition,  $T_{\sigma_k} v_{\sigma_{k-1}} = Tv_{\sigma_{k-1}} \geq T_{\sigma_{k-1}} v_{\sigma_{k-1}} = v_{\sigma_{k-1}}$ . Applying  $T_{\sigma_k}$  on both sides repeatedly, since  $T_{\sigma_k}$  is order-preserving, iteration yields  $T_{\sigma_k}^n v_{\sigma_{k-1}} \geq T_{\sigma_k} v_{\sigma_{k-1}} \geq Tv_{\sigma_{k-1}} \geq v_{\sigma_{k-1}}$ . Taking  $n$  to infinity, the global stability of the policy operator implies that  $v_{\sigma_k} \geq Tv_{\sigma_{k-1}} \geq v_{\sigma_{k-1}}$  for all  $k \geq 0$ . Since  $T$  is order-preserving, we have  $Tv_{\sigma_k} \geq T^2 v_{\sigma_{k-1}}$  for all  $k \in \mathbb{N}$  and then  $v_{\sigma_k} \geq Tv_{\sigma_{k-1}} \geq T^2 v_{\sigma_{k-2}}$ . Induction yields  $v_{\sigma_k} \geq T^k v_{\sigma_0}$  for  $k \in \mathbb{N}$ . Since  $v^* \geq v_{\sigma_k}$  by definition, we obtain

$v^* \geq v_{\sigma_k} \geq T^k v_{\sigma_0}$ . Taking  $k \rightarrow \infty$ , since  $T$  is globally stable with unique fixed point  $v^*$ , we have  $v_{\sigma_k} \rightarrow v^*$  as  $k \rightarrow \infty$ .

Next, let  $\sigma^*$  be  $v^*$  greedy policy. Since  $T_{\sigma^*} v^* = T v^* = v^*$  and  $T_{\sigma^*}$  has a unique fixed point  $v_{\sigma^*}$ , we have  $v_{\sigma^*} = v^*$  and  $H v^* = v_{\sigma^*} = v^*$ , whence  $v^*$  is a fixed point of  $H$ . To see that  $v^*$  is the unique fixed point of  $H$ , let  $\bar{v}$  be the fixed point of  $H$ . Then, we have  $\bar{v} = H \bar{v} = v_\sigma$  where  $\sigma$  is  $\bar{v}$ -greedy. It implies that  $T \bar{v} = T v_\sigma = T_\sigma v_\sigma = v_\sigma = \bar{v}$ . Since  $v^*$  is the unique fixed point of  $T$  by global stability, we have  $\bar{v} = v^*$ .  $\square$

**Lemma A.11.** *Let  $V_u := \{v \in \text{cbX} : Tv \geq v\}$ . If Condition 2.2 holds, then  $W$ , defined by (8), is an order-preserving self-map on  $V_u$ , and we have*

$$v \in V_u \implies Tv \leq Wv \leq T^m v.$$

*Proof of Lemma A.11.* Let Condition 2.2 hold. Then,  $T$  and  $T_\sigma$  are order-preserving self-maps on  $\text{cbX}$  for all  $\sigma \in \Sigma_C$ , and  $v$ -greedy continuous policy exists for all  $v \in \text{cbX}$  by Lemma A.8. Fix any  $v \in V_u$  and let  $\sigma \in \Sigma_C$  be such that  $T_\sigma v = Tv$ . Since  $T_\sigma u \leq Tu$  for any  $u \in \text{cbX}$ ,  $T$  and  $T_\sigma$  are order-preserving and  $v \leq Tv$ , we see that  $Wv \in V_u$ :

$$Wv = T_\sigma T_\sigma^{m-1} v \leq T T_\sigma^{m-1} v \leq T T_\sigma^{m-1} Tv = T T_\sigma^{m-1} T_\sigma v = TWv.$$

Then,  $W$  is a self-map on  $V_u$ . Also since  $T_\sigma^m$  is order-preserving,  $W$  is order-preserving. Next, regarding the first inequality, since  $T_\sigma$  is order-preserving,  $v \leq Tv$ , and  $T_\sigma v = Tv$ , we have

$$T_\sigma^{m-1} v \leq T_\sigma^{m-1} Tv = T_\sigma^{m-1} T_\sigma v = Wv.$$

Repeating the same iteration, we have  $T_\sigma^{m-j} v \leq Wv$  for  $j < m$ . In particular, for  $j = m - 1$ , we have  $T_\sigma v \leq Wv$ . Since  $T_\sigma v = Tv$ , we obtain  $Tv \leq Wv$ . For the second inequality, since  $T_\sigma v \leq Tv$  by definition, and  $T$  and  $T_\sigma$  are order-preserving, we have  $Wv = T_\sigma^m v \leq T^m v$ .  $\square$

**Lemma A.12 (OPI).** *If Condition 2.2 holds and  $T$  is globally stable on  $\text{cbX}$  with unique fixed point  $v^* \in \text{cbX}$ , then the OPI iteration  $\{v_k\}$  converges to  $v^*$  with  $v_0 = v_\sigma$  for some  $\sigma \in \Sigma_C$ .*

*Proof of Lemma A.12.* Let all the stated assumptions hold. Pick  $\sigma \in \Sigma_C$  and let  $v_0 = v_\sigma$ . Let  $\{v_k\}_{k \geq 0}$  be the OPI iteration. First, claim that

$$T^k v_0 \leq W^k v_0 \leq T^{km} v_0 \quad (k \in \mathbb{N}).$$

Since  $v_0 = T_\sigma v_0 \leq T v_0$ ,  $v_0 \in V_u$ . Hence, it follows from Lemma A.11 that the claim holds for  $k = 1$ . Suppose that the claim holds for some  $k \in \mathbb{N}$ . Since all operators are order-preserving and self-map on  $V_u$ , Lemma A.11 with  $W^k v_0, T^{km} v_0 \in V_u$  implies the iteration:

$$T^{k+1} v_0 \leq T W^k v_0 \leq W W^k v_0 \leq W T^{km} v_0 \leq T^{(k+1)m} v_0.$$

Therefore, the claim holds for all  $k \in \mathbb{N}$  by induction. Now, since  $T^k v \rightarrow v^*$  as  $k \rightarrow \infty$ , the above claim implies that  $W^k v_0 \rightarrow v^*$ . Since OPI iteration follows  $v_k = W^k v_0$ , we conclude that  $\{v_k\}$  converges to  $v^*$ .  $\square$

*Proof of Theorem 2.1.* Let  $\mathcal{R}$  be regular. If Assumption 2.1 holds, then Lemma A.1 implies (a), and Proposition A.1 shows the other statements. If Assumption 2.2 holds, then Lemma A.2 implies (a), and Lemma A.3 implies that  $T$  is globally stable. Moreover, Lemma A.4 implies (b). If Assumption 2.3 holds, then Lemma A.6 shows that  $T$  and  $T_\sigma$  are eventually contracting on  $cb\mathbf{X}$  and  $mb\mathbf{X}$ , respectively, for all  $\sigma \in \Sigma$ . Suppose that either Assumption 2.2, or 2.3 holds. Part (c), (d), and (g) follow from Lemma A.7 with global stability of  $T$ . Part (f) follows from Lemma A.9 with (c). Part ( $\alpha$ ) follows from Lemma A.10 with (c). Part ( $\beta$ ) follows from Lemma A.12 with (c).  $\square$

*Proof of Proposition 2.1.* Let the stated assumptions hold. Fix  $v, w \in U \subset mb\mathbf{X}$ . Then, by the properties of  $\mathcal{T}$ , we have

$$\mathcal{T}v = \mathcal{T}(v + w - w) \leq \mathcal{T}(w + |v - w|) \leq \mathcal{T}w + G|v - w|.$$

Exchanging the roles of  $v$  and  $w$ , we obtain  $|\mathcal{T}v - \mathcal{T}w| \leq G|v - w|$ . Then, by the same proof in Lemma A.6,  $\mathcal{T}$  is eventually contracting on  $U$ .  $\square$

**Lemma A.13.** *If Condition 2.2 holds,  $T_\sigma$  is globally stable on  $mb\mathbf{X}$  for all  $\sigma \in \Sigma$ , and  $T$  is globally stable on  $cb\mathbf{X}$ , then  $v^* = \sup_{\sigma \in \Sigma_C} v_\sigma$  and  $Tv = \sup_{\sigma \in \Sigma_C} T_\sigma v$  for all  $v \in cb\mathbf{X}$ .*

*Proof of Lemma A.13.* Let all the assumptions hold. Fix  $v \in cb\mathbf{X}$ . By Lemma A.8, there is a continuous  $\sigma' \in \Sigma$  such that  $Tv = T_{\sigma'} v$ , which implies that  $Tv = \sup_{\sigma \in \Sigma} T_\sigma v = T_{\sigma'} v \leq \sup_{\sigma \in \Sigma_C} T_\sigma v$ . Moreover, since Lemma A.7 implies  $v^* = \bar{v} \in cb\mathbf{X}$ , we have  $\sup_{\sigma \in \Sigma} v_\sigma = v^* = v_{\sigma^*} \leq \sup_{\sigma \in \Sigma_C} v_\sigma$ , where  $\sigma^*$  is a continuous  $v^*$ -greedy policy.  $\square$

### A.2. Proofs in Section 3.

*Proof of Lemma 3.1.* Let  $\mathcal{M}$  be an MDP satisfying Condition 3.1. Since  $(x, a) \mapsto B(x, a, v) = r(x, a) + \int_{\mathbf{X}} v(x') \beta(x, a, x') P(x, a, dx')$  is continuous and bounded on  $\mathbf{G}$  for  $v \in cb\mathbf{X}$  by assumption, Condition 2.1 holds. Clearly, the value aggregator  $B$  satisfies monotonicity condition (2). Therefore,  $\mathcal{M}$  is regular. Lemma 2.1 concludes the remaining statements.  $\square$

**Lemma A.14.** *If Condition 3.2 holds, then the following statements are true.*

- (a)  $T$  and  $T_\sigma$  are self-maps on  $cb\mathbf{X}$ , and
- (b) for any  $v \in cb\mathbf{X}$  there exists a continuous  $v$ -greedy policy.

*Proof of Lemma A.14.* The statements follow from Lemma A.8 that Condition 3.2 holds implies Condition 2.2.  $\square$

*Proof of Lemma 3.2.* Let  $\mathcal{M}$  be regular and Assumption 3.3 hold. Then, we have  $L_\sigma \mathbb{1} \leq L \mathbb{1}$  for all  $\sigma \in \Sigma$ . Pick  $\sigma \in \Sigma$ . Since  $L_\sigma$  and  $L$  are order-preserving, iteration implies  $L_\sigma^n \mathbb{1} \leq L^n \mathbb{1}$  for all  $n \in \mathbb{N}$ . Let  $\{X_t\}$  be a  $P_\sigma$ -Markov process with  $X_0 = x$  and  $\beta_t = \beta(X_t, \sigma(X_t), X_{t+1})$  for all  $t \in \mathbb{N}_0$ . Then, iteration implies

$$L_\sigma^n \mathbb{1}(x) = \mathbb{E}_x^\sigma \{\beta_0 \beta_1 \cdots \beta_{n-1}\} \leq L^n \mathbb{1}(x) \leq \|L^n \mathbb{1}\|$$

for  $x \in \mathbf{X}$ . Finally, taking supremum over  $\mathbf{X}$ , we have  $d_n^\sigma = \|L_\sigma^n \mathbb{1}\| \leq \|L^n \mathbb{1}\|$  for all  $n \in \mathbb{N}$ . Since  $\lim_{n \rightarrow \infty} \|L^n \mathbb{1}\|^{1/n} = \rho(L) < 1$ , there exists a  $n \in \mathbb{N}$  satisfying  $\|L^n \mathbb{1}\| < 1$  and then  $d_n^\sigma < 1$  for all  $\sigma \in \Sigma$ . Then, we have

$$\rho(L_\sigma) = \lim_{n \rightarrow \infty} \|L_\sigma^n \mathbb{1}\|^{1/n} \leq \lim_{n \rightarrow \infty} \|L^n \mathbb{1}\|^{1/n} = \rho(L) < 1.$$

Now, since  $\hat{L} \mathbb{1} = \sup_\sigma L_\sigma \mathbb{1} \leq L \mathbb{1}$ , iteration yields  $\hat{L}^n \mathbb{1} \leq L^n \mathbb{1}$  for  $n \in \mathbb{N}$ . Then, the same argument implies  $\rho(\hat{L}) \leq \rho(L) < 1$ .  $\square$

**Lemma A.15.** *If  $\mathcal{M}$  is regular and Assumption 3.2 holds, then  $T_\sigma$  is eventually contracting and globally stable on  $mb\mathbf{X}$ , and  $v_\sigma$  defined by (10) is the unique fixed point of  $T_\sigma$  in  $mb\mathbf{X}$  for any  $\sigma \in \Sigma$ .*

*Proof of Lemma A.15.* Suppose that  $\mathcal{M}$  is regular and Assumption 3.2 holds. Then, Assumption 2.2 holds. Fix  $\sigma \in \Sigma$ . Then, it follows from Lemma A.2 and Lemma 2.2 that  $T_\sigma$  is globally stable and eventually discounting and has a unique fixed point.

Next, we check that  $v_\sigma$ , defined by (10), is a fixed point of  $T_\sigma$  by definition. Let  $\{X_t\}_{t \geq 0}$  be a stochastic process generated by  $P_\sigma$  with  $X_0 = x$ . Let  $\beta_t = \beta(X_t, \sigma(X_t), X_{t+1})$  for  $t \in \mathbb{N}_0$ . Since  $v_s = T_\sigma v_s = T_\sigma^n v_s$ , we have

$$\begin{aligned}
v_s(x) &= r_\sigma(x) + \mathbb{E}_x^\sigma \beta_0 v_s(X_1) \\
&= r_\sigma(x) + \mathbb{E}_x^\sigma \beta_0 [r_\sigma(X_1) + \mathbb{E}_{X_1}^\sigma \beta_1 v_s(X_2)] \\
&= \dots \\
&= \mathbb{E}_x^\sigma \left[ \sum_{t=0}^n \prod_{i=0}^{t-1} \beta_i r_\sigma(X_t) \right] + \mathbb{E}_x^\sigma \beta_0 \beta_1 \cdots \beta_n v_s(X_{n+1}) \\
&= \lim_{n \rightarrow \infty} \mathbb{E}_x^\sigma \left[ \sum_{t=0}^n \prod_{i=0}^{t-1} \beta_i r_\sigma(X_t) \right] = v_\sigma(x), \quad (x \in \mathbf{X})
\end{aligned}$$

where we use

$$\begin{aligned}
\lim_{n \rightarrow \infty} \mathbb{E}_x^\sigma \beta_0 \beta_1 \cdots \beta_n v_s(X_{n+1}) &\leq \lim_{n \rightarrow \infty} \left( \sup_x \mathbb{E}_x^\sigma \beta_0 \beta_1 \cdots \beta_n \|v_s\| \right) \\
&\leq \lim_{\substack{n \rightarrow \infty, \\ n = kn_\sigma + i, \\ i < n_\sigma, i, k \in \mathbb{N}}} \left( \sup_x \mathbb{E}_x^\sigma \prod_{t=0}^{kn_\sigma-1} \beta_t \mathbb{E}_{X_{kn_\sigma}}^\sigma \prod_{j=kn_\sigma}^{kn_\sigma+i-1} \beta_j \|v_s\| \right) \\
&\leq \left( \lim_{k \rightarrow \infty} (d_{n_\sigma}^\sigma)^k \right) \sup_{i < n_\sigma} d_i^\sigma \|v_s\| = 0.
\end{aligned}$$

Therefore,  $v_s = v_\sigma$ . □

**Lemma A.16.** *If  $\mathcal{M}$  is regular, and either Assumption 3.1, 3.2, or 3.3 holds, then  $T_\sigma$  is contractive on  $(mb\mathbf{X}, \|\cdot\|_{w_\sigma})$  with modulus  $\lambda_\sigma$ , and  $T$  is contractive on  $(cb\mathbf{X}, \|\cdot\|_w)$  with modulus  $\lambda$ , where*

$$w_\sigma(x) := \mathbb{E}_x^\sigma \sum_{t=0}^{\infty} \prod_{i=0}^{t-1} \beta_i^\sigma \quad \text{and} \quad \lambda_\sigma := \sup_{x \in \mathbf{X}} \frac{w_\sigma(x) - 1}{w_\sigma(x)}$$

for  $x \in \mathbf{X}$  and  $w := \sum_{n=0}^{\infty} \hat{L}^n \mathbb{1}$  and  $\lambda := \sup_{x \in \mathbf{X}} (w(x) - 1)/w(x)$ . Moreover,  $L_\sigma w_\sigma \leq \lambda_\sigma w_\sigma$  for all  $\sigma \in \Sigma$  and  $\hat{L}w \leq \lambda w$ .

*Proof of Lemma A.16.* Let  $\mathcal{M}$  be regular and either Assumption 3.1, 3.2, or 3.3 holds. Then, assumptions of Theorem 2.1 hold. Hence, given any reward function satisfying regular conditions,  $T$  is globally stable on  $(cb\mathbf{X}, \|\cdot\|)$ , and  $T_\sigma$  is globally stable on  $(mb\mathbf{X}, \|\cdot\|)$  and has a unique fixed point  $v_\sigma$ , for all  $\sigma \in \Sigma$ . Fix  $\sigma \in \Sigma$ . If we consider  $r \equiv \mathbb{1}$ , then the policy value is  $w_\sigma(x) = \mathbb{E}_x^\sigma \sum_{t=0}^{\infty} \prod_{i=0}^{t-1} \beta_i^\sigma$ , where

$\beta_t^\sigma = \beta_\sigma(X_t, X_{t+1})$ , given a stochastic process  $\{X_t\}$  generated by  $P_\sigma$  with  $X_0 = x$ . Since  $\beta$  is bounded and strictly positive, and  $\prod_{i=1}^{-1} \beta_i^\sigma = 1$ , we have  $w_\sigma \gg \mathbb{1}$ . Since  $w_\sigma$  is the fixed point of  $T_\sigma$  when  $r = \mathbb{1}$ ,  $w_\sigma$  is bounded. Therefore, we have

$$w_\sigma(x) = 1 + \int w_\sigma(x') \beta_\sigma(x, x') P_\sigma(x, dx') \quad (x \in \mathbf{X}).$$

Rewriting this equation, we have for all  $x \in \mathbf{X}$

$$\begin{aligned} L_\sigma w_\sigma(x) &= \int w_\sigma(x') \beta_\sigma(x, x') P_\sigma(x, dx') \\ &= w_\sigma(x) - 1 \leq \left( \sup_y \frac{w_\sigma(y) - 1}{w_\sigma(y)} \right) w_\sigma(x) =: \lambda_\sigma w_\sigma(x). \end{aligned}$$

Since  $w_\sigma$  is bounded and  $w_\sigma \gg \mathbb{1}$ , we obtain  $\lambda_\sigma < 1$ . Therefore, fixing  $v, v' \in mb\mathbf{X}$ , we have

$$\begin{aligned} |T_\sigma(x)v - T_\sigma v'(x)| &\leq L_\sigma |v - v'| (x) \leq L_\sigma (\|v - v'\|_{w_\sigma} w_\sigma)(x) \\ &\leq \|v - v'\|_{w_\sigma} \lambda_\sigma w_\sigma(x) \end{aligned}$$

for all  $x \in \mathbf{X}$ . Dividing  $w_\sigma(x)$  and taking supremum over  $\mathbf{X}$ , we have  $\|T_\sigma v - T_\sigma v'\|_{w_\sigma} \leq \lambda_\sigma \|v - v'\|_{w_\sigma}$ , so  $T_s$  is contractive with modulus  $\lambda_\sigma$ . Similarly, since  $T$  is globally stable on  $(cb\mathbf{X}, \|\cdot\|)$ ,  $T$  admits a fixed point  $w \in cb\mathbf{X}$  when  $r \equiv \mathbb{1}$ . Then, we have

$$w = \sup_\sigma (\mathbb{1} + L_\sigma w) = \mathbb{1} + \sup_\sigma L_\sigma w = \mathbb{1} + \hat{L}w = \cdots = \sum_{n=0}^{\infty} \hat{L}^n \mathbb{1}.$$

where the last equality follows from the iteration of  $w = \mathbb{1} + \hat{L}w$ . It implies that  $\lambda := \sup_{x \in \mathbf{X}} (w(x) - 1)/w(x) < 1$  and  $\hat{L}w = w - \mathbb{1} \leq \lambda w$ . Fix  $v, v' \in cb\mathbf{X}$ . We have

$$|Tv - Tv'| \leq \hat{L}|v - v'| \leq \hat{L}(\|v - v'\|_w w) \leq \|v - v'\| \lambda w,$$

which shows that  $\|Tv - Tv'\|_w \leq \lambda \|v - v'\|_w$  for all  $v, v' \in cb\mathbf{X}$ .  $\square$

**Lemma A.17.** *If  $\mathcal{R}$  is regular,  $r(x, a) \equiv f(x) > 0$  for all  $x \in \mathbf{X}$  and some  $f \in cb\mathbf{X}_+$ , and  $T$  has a fixed point  $v \in cb\mathbf{X}$ , then  $\rho(\hat{L}) < 1$  and  $\sup_x d_n(x) < 1$ .*

*Proof of Lemma A.17.* Suppose that  $\mathcal{R}$  is regular,  $r \equiv f \in cb\mathbf{X}_+$ , and  $T$  has a fixed point  $v \in cb\mathbf{X}$ . Since  $v$  is the fixed point of  $T$ , we have

$$v = Tv = f + \hat{L}v = f + \hat{L}f + \hat{L}^2 f = \cdots = \sum_{n=0}^{\infty} \hat{L}^n f.$$

Since  $\sum_{n=0}^{\infty} \hat{L}^n f$  converges to  $v$ , it must be that  $\lim_{n \rightarrow \infty} \hat{L}^n f = 0$ . To this end, we have  $\rho(\hat{L}) = \lim_{n \rightarrow \infty} \|\hat{L}^n f\|^{1/n} < 1$ . Since  $\rho(\hat{L}) < 1$ , we have  $\lim_{n \rightarrow \infty} \hat{L}^n \mathbb{1} = 0$ . Therefore, since iteration yields  $d_n = \hat{L}^n \mathbb{1}$  for  $n \in \mathbb{N}$ , there is  $n \in \mathbb{N}$  such that  $\sup_x d_n(x) = \|\hat{L}^n \mathbb{1}\| < 1$ .  $\square$

*Proof of Theorem 3.1.* Let  $\mathcal{M}$  be regular. Lemma 3.1 implies that  $\mathcal{M}$  is a regular RDP. The first statement follows from A.17. Suppose that either Assumption 3.1, 3.2 or 3.3 holds. Lemma A.16 implies that  $T$  (resp.  $T_\sigma$ ) is contractive on  $(cb\mathbf{X}, \|\cdot\|_w)$  (resp.  $(mb\mathbf{X}, \|\cdot\|_{w_\sigma})$ ) with modulus  $\lambda$  (resp.  $\lambda_\sigma$ ). Since Assumption 3.1 implies Assumption 2.1, Assumption 3.2 implies Assumption 2.2, Assumption 3.3 implies Assumption 2.3, and Condition 3.2 implies Condition 2.2, the remaining statements follow from Theorem 2.1.  $\square$

**A.3. Proofs in Section 4.** Let  $\sigma \in \Sigma_C$ . Let  $\{X_t^\sigma\}_{t \geq 0}$  be a  $k_\sigma$ -Markov process. Suppose that  $\{X_t^\sigma\}_{t \geq 0}$  admits a stationary distribution  $\pi_\sigma$ . Let  $L_1(\pi_\sigma)$  be the set of Borel measurable functions  $g: \mathbf{X} \rightarrow \mathbb{R}$  such that

$$\|g\| := \int |g(x)| \pi_\sigma(dx) < \infty.$$

For  $f, g \in L_1(\pi_\sigma)$ , we write  $f \geq g$  if  $f(x) \geq g(x)$  for  $\pi_\sigma$ -almost all  $x \in \mathbf{X}$ . We write  $f \gg g$  if  $f(x) > g(x)$  for  $\pi_\sigma$ -almost all  $x \in \mathbf{X}$ . Define  $\mathcal{G}_\sigma$  to be all  $f \in L_1(\pi_\sigma)$  such that  $f \gg 0$ .

*Proof of Lemma 4.1.* The proof is identical to Lemma A.14 with the fact that  $r$  is positive everywhere.  $\square$

**Lemma A.18.** *If Assumption 4.1 and Condition 4.1 hold, then for all  $\sigma \in \Sigma_C$  the following statements are true.*

- (a)  $k_\sigma$  is continuous and bounded,
- (b) there exists an  $m \in \mathbb{N}$  such that  $k_\sigma^m$  is everywhere positive,
- (c) there exists a unique stationary density  $\pi_\sigma$  for  $k_\sigma$  on  $\mathbf{X}$ , and
- (d)  $\pi_\sigma$  is everywhere positive and continuous on  $\mathbf{X}$

*Proof of Lemma A.18.* Let the stated assumptions hold. Fix  $\sigma \in \Sigma_C$ . Since  $\beta$ ,  $P$  and  $\sigma$  are continuous,  $\beta_\sigma$  and  $P_\sigma$  are continuous on a compact set  $\mathbf{X}$  so that they are

bounded. Then,  $k_\sigma$  is continuous and bounded, which shows (a). Let  $\{X_t^\sigma\}_{t \geq 0}$  be the state process such that

$$\mathbb{P}\{X_{t+1}^\sigma \in B | X_t^\sigma = x\} = \int_B k_\sigma(x, dy)$$

for every  $x \in \mathsf{X}$  and Borel set  $B \subset \mathsf{X}$ . Since Assumption 4.1 holds and  $\beta$  is strictly positive, the Markov chain induced by  $P_\sigma$  is irreducible, which implies that the Markov chain induced by  $k_\sigma$  is also irreducible. Hence, there exists an  $m \in \mathbb{N}$  such that  $k_\sigma^m$  is everywhere positive, which shows (b), and then  $\{X_t^\sigma\}$  is irreducible. It then implies that (c): there exists a unique stationary distribution  $\pi_\sigma$ :  $X_t^\sigma \stackrel{d}{=} \pi_\sigma$ . Finally it follows from the proof of Lemma C1 of Borovička and Stachurski (2020) that (d):  $\pi_\sigma$  is everywhere positive and continuous.  $\square$

**Lemma A.19.** *If Assumption 4.1 and Condition 4.1 hold, then the following statements are true for all  $\sigma \in \Sigma_C$ :*

- (a)  $L_\sigma$  is a bounded linear operator on  $L_1(\pi_\sigma)$ .
- (b)  $L_\sigma g$  is continuous for  $g \in \mathcal{G}_\sigma$ .
- (c)  $L_\sigma g \geq 0$  when  $g \geq 0$  and  $L_\sigma g \in \mathcal{G}_\sigma$  when  $g \in \mathcal{G}_\sigma$ .
- (d)  $L_\sigma$  is irreducible and  $L_\sigma^2$  is compact.
- (e)  $\rho(L_\sigma) > 0$  and there exists a continuous function  $e_\sigma \in \mathcal{G}_\sigma$  such that  $L_\sigma e_\sigma = \rho(L_\sigma) e_\sigma$ .
- (f)  $L_\sigma$  is order preserving on  $L_1(\pi_\sigma)$ .

*Proof of Lemma A.19.* Let Assumption 4.1 and Condition 4.1 hold. Fix  $\sigma \in \Sigma_C$ . For (a) and (b), since  $k_\sigma$  is continuous and bounded by Lemma A.18, and  $\bar{\beta}_\sigma$  is bounded and everywhere positive, the result follows from the proof in Lemma C2 of Borovička and Stachurski (2020). For (c), the first claim is obvious, and the second follows from that  $\beta$  is everywhere positive. For (d), the proof is identical to Lemma C3 of Borovička and Stachurski (2020) with continuity of  $\pi_\sigma$  and  $k_\sigma$  from Lemma A.18. Part (e) and (f) are identical to Lemma 8.4 of Stachurski et al. (2022a).  $\square$

**Lemma A.20.** *If Assumption 4.1 and Condition 4.1 hold, then  $\|T_\sigma v - T_\sigma w\|_{e_\sigma} \leq \rho(L_\sigma) \|v - w\|_{e_\sigma}$  for all  $v, w \in cb\mathsf{X}_+$  and all  $\sigma \in \Sigma_C$ .*



*Proof of Lemma A.20.* Let Assumption 4.1 and Condition 4.1 hold. Fix  $\sigma \in \Sigma_C$ . Lemma A.19 implies that there is  $e_\sigma \in cb\mathbf{X}_+$  such that  $L_\sigma e_\sigma = \rho(L_\sigma)e_\sigma$ . Then, for  $v, w \in cb\mathbf{X}_+$  we have

$$\begin{aligned}
|T_\sigma v(x) - T_\sigma w(x)| &= \left| \bar{\beta}_\sigma(x) \int k_\sigma(x, dx') (v(x') - w(x')) \right| \\
&\leq \bar{\beta}_\sigma(x) \int k_\sigma(x, dx') |v(x') - w(x')| \\
&\leq \bar{\beta}_\sigma(x) \int k_\sigma(x, dx') e_\sigma(x') \sup_{x'} \frac{|v(x') - w(x')|}{e_\sigma(x')} \\
&= \bar{\beta}_\sigma(x) \int k_\sigma(x, dx') e_\sigma(x') \|v - w\|_{e_\sigma} \\
&= \|v - w\|_{e_\sigma} L_\sigma e_\sigma(x) = \|v - w\|_{e_\sigma} \rho(L_\sigma) e_\sigma(x).
\end{aligned} \tag{26}$$

Dividing both sides with  $e_\sigma(x)$  and taking the supremum, we have  $\|T_\sigma v - T_\sigma w\|_{e_\sigma} \leq \rho(L_\sigma) \|v - w\|_{e_\sigma}$  for all  $v, w \in cb\mathbf{X}_+$ .  $\square$

*Proof of Proposition 4.1.* Let Assumption 4.1 and Condition 4.1 hold. Part (a) and (b) are equivalent by Lemma 2.2. Fix  $\sigma \in \Sigma_C$ . Then, we have  $r_\sigma \in cb\mathbf{X}_+$  and  $T_\sigma v = r_\sigma + L_\sigma v$  for  $v \in cb\mathbf{X}_+$ . Since  $L_\sigma$  is irreducible and  $L_\sigma^2$  is compact by Lemma A.19, it follows from Theorem 3.1 of Stachurski et al. (2022b) that part (a) and (c) are equivalent, and  $T_\sigma$  has no fixed point in  $cb\mathbf{X}_+$  if  $\rho(L_\sigma) \geq 1$ . Then, part (e) and (f) implies part (a), and clearly part (c) implies (e) and (f). Suppose that  $\rho(L_\sigma) < 1$ . Lemma A.20 shows that  $\|T_\sigma v - T_\sigma w\|_{e_\sigma} \leq \rho(L_\sigma) \|v - w\|_{e_\sigma}$  for all  $v, w \in cb\mathbf{X}_+$ , whence a contraction map so that (a) implies (d). Finally, if that  $T_\sigma$  is a contraction map in  $\|\cdot\|_{e_\sigma}$ , then it is globally stable.  $\square$

**Lemma A.21.** *If Condition 4.1 holds and  $\rho(\bar{L}) < 1$ , then  $T$  is eventually contracting, where  $\bar{L}v := \sup_{\sigma \in \Sigma_C} L_\sigma v$  for  $v \in cb\mathbf{X}_+$ . Moreover,  $T$  is contracting on  $(cb\mathbf{X}, \|\cdot\|_w)$  with modulus  $\lambda$ , where  $w := \sum_{n=0}^{\infty} \bar{L}^n \mathbb{1}$  and  $\lambda = \sup_x (w(x) - 1)/w(x)$ .*

*Proof of Lemma A.21.* Suppose that Condition 4.1 holds and  $\rho(\bar{L}) < 1$ . Since for all  $v \in cb\mathbf{X}_+$ , there exists a  $v$ -greedy policy  $\sigma \in \Sigma_C$ , we have  $Tv = T_\sigma v \leq \sup_{\sigma \in \Sigma_C} T_\sigma v$ . Since  $Tv \geq \sup_{\sigma \in \Sigma_C} T_\sigma v$  by definition, we have  $Tv = \sup_{\sigma \in \Sigma_C} T_\sigma v$ . Then, the similar proof for Lemma A.3 shows that  $\|T^n v - T^n v'\| \leq \|\bar{L}^n \mathbb{1}\| \|v - v'\|$  for any  $v, v' \in cb\mathbf{X}_+$ . Since  $\rho(\bar{L}) < 1$ , there is  $m \in \mathbb{N}$  such that  $\|\bar{L}^m \mathbb{1}\| \leq 1$ . Then,  $T^m$  is contractive. Finally, the method in the proof of Lemma A.16 shows that  $T$  is contracting on  $(cb\mathbf{X}, \|\cdot\|_w)$  with modulus  $\lambda$ .  $\square$

*Proof of Theorem 4.1.* Let Condition 4.1 hold. Lemma A.13 implies  $Tv = \sup_{\sigma \in \Sigma_C} T_\sigma v$  for  $v \in cbX_+$ . If  $r \equiv f \in cbX_+$ , and  $T$  has a fixed point in  $cbX_+$ , then the argument in Lemma A.17, with  $\bar{L}$  instead of  $\hat{L}$ , shows  $\rho(\bar{L}) < 1$ . Suppose that Assumption 4.1 and 4.2 hold. Proposition 4.1 implies (a). Lemma A.21 implies (b). The remaining statements follow from Theorem 3.1.  $\square$

**A.4. Proofs in Section 6.** We also consider the following regular conditions in this section.

**Condition A.1.**

- (i)  $\Gamma$  is nonempty, compact-valued.
- (ii)  $a \mapsto r(x, a)$  is u.s.c. for all  $x \in X$ .
- (iii)  $\beta$  is bounded and strictly positive.
- (iv)  $a \mapsto \int_X f(y)\beta(x, a, y)P(x, a, y)dy$  is continuous on  $\Gamma(x)$  for all  $x \in X$  and for  $f \in mbX$ .

If Condition A.1 is satisfied, we consider the value function spaces  $\mathcal{V}$  and  $\mathcal{G}$  defined by

$$\begin{aligned} \mathcal{V} &:= \{v: X \rightarrow \mathbb{R} \cup \{-\infty\}: v \in mX, v/\kappa \text{ is bounded above,} \\ &\quad \text{and } (x, a) \mapsto \mathbb{E}_{x,a} v(x') \text{ is bounded below}\}, \\ \mathcal{G} &:= \{g: G \rightarrow \mathbb{R}: g \in mG, g \text{ is bounded below, } \|g\|_\kappa < \infty, \\ &\quad \text{and } a \mapsto g(x, a) \text{ is u.s.c. on } \Gamma(x) \text{ for all } x \in X\} \end{aligned}$$

where  $\kappa \geq 1$  is a real-valued function on  $X$ , which is further defined in Assumption 6.1. We say that an MDP is *regular* if either one of Condition A.1 or 6.1 holds.

**Lemma A.22.** *Suppose that Assumption 6.1 holds. If  $\mathcal{M}$  is regular, then  $(\mathcal{G}, \|\cdot\|_\kappa)$  is a Banach space.*

*Proof of Lemma A.22.* Suppose that Assumption 6.1 holds, and  $\mathcal{M}$  is regular. If Condition 6.1 holds, then  $(\mathcal{G}, \|\cdot\|_\kappa)$  is complete follows from Ma et al. (2022). Suppose that Condition A.1 holds. Let  $B_\kappa(G)$  be the space of Borel measurable real-valued functions  $f$  on  $G$  satisfying  $\|f\|_\kappa < \infty$ . Since  $(B_\kappa(G), \|\cdot\|_\kappa)$  is a Banach space, it suffices to show that  $\mathcal{G}$  is closed in  $B_\kappa(G)$ . Let  $\{g_n\} \subset \mathcal{G}$  such that  $\|g_n - g\|_\kappa \rightarrow 0$ . Clearly,  $\|g\|_\kappa$  is finite. We next show that  $a \mapsto g(x, a)$  is u.s.c. for all  $x \in X$ . Fix

$x' \in \mathbf{X}$ . For all  $a_0 \in \Gamma(x')$  and  $y > g(x', a_0)$ , let  $\varepsilon = y - g(x', a_0)$ . Since  $\|g_n - g\|_\kappa \rightarrow 0$ , for all  $\delta > 0$  there exist  $N \in \mathbb{N}$  such that for all  $x$  and  $a \in \Gamma(x)$  we have

$$|g_N(x, a) - g(x, a)| < \kappa(x)\delta.$$

We choose  $\delta$  such that  $\kappa(x')\delta < \varepsilon/3$ . Since  $g_N(x, \cdot)$  is u.s.c. on  $G(x)$  for all  $x \in \mathbf{X}$ , there exists a neighborhood  $U$  of  $a_0$  such that for all  $a \in U$

$$g_N(x', a) < g_N(x', a_0) + \varepsilon/3.$$

Hence, the previous inequalities imply

$$\begin{aligned} g(x', a) &< g_N(x', a) + \kappa(x')\delta < g_N(x', a_0) + \kappa(x')\delta + \varepsilon/3 \\ &< g(x', a_0) + 2\kappa(x')\delta + \varepsilon/3 < g(x', a_0) + \varepsilon < y. \end{aligned}$$

for all  $a \in U$ . Therefore,  $a \rightarrow g(x', a)$  is u.s.c.. Since  $x'$  is arbitrarily picked,  $a \mapsto g(x, a)$  is u.s.c. on  $\Gamma(x)$  for all  $x \in \mathbf{X}$ . We conclude that  $\mathcal{G}$  is closed in  $B_\kappa(\mathbb{G})$  and then  $(\mathcal{G}, \|\cdot\|_\kappa)$  is complete.  $\square$

**Lemma A.23** (Well-defined Value). *If Assumption 6.1 hold, then  $v_\sigma(x)$  and  $v^*(x)$  are well-defined in  $\mathbb{R} \cup \{-\infty\}$  for all  $x \in \mathbf{X}$  and  $\sigma \in \Sigma$ .*

*Proof of Lemma A.23.* Suppose that Assumption 6.1 holds. Fixing  $x$  and  $\sigma \in \Sigma$ . Since  $\bar{r}(x) \leq d\kappa(x)$  and  $\mathbb{E}_{x,a}\beta(x, a, x')\kappa(x')L^n\mathbb{1}(x') \leq \alpha\kappa(x)L^{n+1}\mathbb{1}(x)$  for all  $x \in \mathbf{X}$  and  $n \geq 0$ , iteration implies that for all  $t \in \mathbb{N}$  we have

$$\begin{aligned} \mathbb{E}_x \prod_{i=0}^{t-1} \beta(x_i, \sigma(x_i), x_{i+1}) r(x_t, \sigma(x_t)) &\leq \mathbb{E}_x \prod_{i=0}^{t-1} \beta(x_i, \sigma(x_i), x_{i+1}) \bar{r}(x_t) \\ &\leq \mathbb{E}_x \prod_{i=0}^{t-1} \beta(x_i, \sigma(x_i), x_{i+1}) d\kappa(x_t) \\ &\leq d \mathbb{E}_x \prod_{i=0}^{t-2} \beta(x_i, \sigma(x_i), x_{i+1}) \alpha\kappa(x_{t-1}) L\mathbb{1}(x_{t-1}) \\ &\leq d\alpha^t \kappa(x) (L^t \mathbb{1})(x). \end{aligned}$$

Since there is an  $n \in \mathbb{N}$  such that  $\alpha^n \|L^n\| < 1$  by assumption, we have

$$\begin{aligned} v_\sigma(x) &= \mathbb{E}_x \sum_{t=0}^{\infty} \prod_{i=0}^{t-1} \beta(x_i, \sigma(x_i), x_{i+1}) r(x_t, \sigma(x_t)) \\ &\leq \sum_{t=0}^{\infty} d\alpha^t \|L^t \mathbb{1}\| \kappa(x) \leq d\kappa(x) \frac{\sum_{t=0}^{n-1} \alpha^t \|L^t\|}{1 - \alpha^n \|L^n\|} < \infty. \end{aligned}$$

Therefore,  $v_\sigma(x)$  is well-defined in  $\mathbb{R} \cup \{-\infty\}$  for all  $x \in \mathbf{X}$  and  $\sigma \in \Sigma$ . Moreover, since the upper bound holds for all  $\sigma \in \Sigma$ , the definition of  $v^*$  implies that  $v^*(x)$  is bounded above and then well-defined for all  $x \in \mathbf{X}$ .  $\square$

**Lemma A.24.** *If  $\mathcal{M}$  is regular and Assumption 6.1 hold, then  $M\mathcal{G} \subset \mathcal{V}$ ,  $EV \subset \mathcal{G}$ ,  $TV \subset \mathcal{V}$  and  $R\mathcal{G} \subset \mathcal{G}$ .*

*Proof of Lemma A.24.* Let Condition A.1 and Assumption 6.1 hold (the proof for Condition 6.1 is similar, see also Ma et al. (2022) for  $R\mathcal{G} \subset \mathcal{G}$ .) We first show that  $EV \subset \mathcal{G}$ . Let  $v \in \mathcal{V}$ . Then,  $v/\kappa$  is bounded above and  $\mathbb{E}_{(\cdot, \cdot)} v(x')$  is bounded below. Similar to Lemma 8.3.7 of Hernández-Lerma and Lasserre (2012b), we can show that  $a \mapsto \mathbb{E}_{x,a} \beta(x, a, x') v(x')$  is u.s.c. on  $\Gamma(x)$  for all  $x \in \mathbf{X}$ . In detail, fix  $x \in \mathbf{X}$  and let  $\{a_n\}_{n \geq 0} \subset \Gamma(x)$  be such that  $a_n \rightarrow a \in \Gamma(x)$ . Since  $v/\kappa$  is bounded above, there exists an  $m \in \mathbb{R}$  such that  $v \leq m\kappa$ . Hence,  $v - m\kappa$  is non-positive, so there is a non-increasing sequence of bounded measurable functions  $\{v_m^k\}$  such that  $v_m^k \downarrow v_m$ .<sup>22</sup> Then, Assumption 6.1 implies that, for all  $k$ ,

$$\begin{aligned} \limsup_{n \rightarrow \infty} \int v_m(x') \beta(x, a_n, x') P(x, a_n, dx') &\leq \limsup_{n \rightarrow \infty} \int v_m^k(x') \beta(x, a_n, x') P(x, a_n, dx') \\ &= \int v_m^k(x') \beta(x, a, x') P(x, a, dx') \end{aligned}$$

Letting  $k \rightarrow \infty$ , the Monotone Convergence theorem yields that

$$\limsup_{n \rightarrow \infty} \int v_m(x') \beta(x, a_n, x') P(x, a_n, dx') \leq \int v_m(x') \beta(x, a, x') P(x, a, dx').$$

To this end, since  $x \in \mathbf{X}$  is arbitrary,  $a \mapsto \mathbb{E}_{x,a} \beta(x, a, X') v_m(X') = Ev_m(x, a)$  is u.s.c. for all  $x \in \Gamma(x)$ . It implies that  $Ev(x, \cdot)$  is u.s.c. on  $\Gamma(x)$ . Furthermore, since  $v/\kappa \leq m$ , Assumption 6.1 implies that, for all  $(x, a) \in \mathbf{G}$ ,

$$Ev(x, a) = \mathbb{E}_{x,a} \beta(x, a, X') v(x') \leq \mathbb{E}_{x,a} \beta(x, a, X') m\kappa(x') \leq m\kappa(x) \alpha L \mathbb{1}(x).$$

<sup>22</sup>If Condition 6.1 holds, then  $\kappa$  is continuous and  $v$  is u.s.c., implying  $v - m\kappa$  is u.s.c.. Then, there is a  $\{v_m^k\} \subset cb\mathbf{X}$  such that  $v_m^k \downarrow v_m$ .

Hence,  $Ev/\kappa$  is bounded above by  $m\alpha\|L\|$ . Moreover, to see that  $Ev$  is bounded below, let  $B := \{x \in \mathbf{X} : v(x) < 0\}$ . Since  $v/\kappa$  is bounded above, we have

$$\int_{\mathbf{X} \setminus B} v(x')P(x, a, dx') \leq \int_{\mathbf{X} \setminus B} m\kappa(x')P(x, a, dx') \leq m\|\kappa\|.$$

for all  $(x, a) \in \mathbf{G}$ . Since  $\mathbb{E}_{(\cdot, \cdot)}v(x')$  is bounded below, there is  $e \in \mathbb{R}$  such that

$$\int_B v(x')P(x, a, dx') + \int_{\mathbf{X} \setminus B} v(x')P(x, a, dx') \geq e$$

for all  $(x, a) \in \mathbf{G}$ . The above two inequalities imply

$$\int_B v(x')P(x, a, dx') \geq e - \int_{\mathbf{X} \setminus B} v(x')P(x, a, dx') \geq e - m\|\kappa\|,$$

for all  $(x, a) \in \mathbf{G}$ . Since  $\beta$  is bounded, we see that  $Ev(\cdot, \cdot) = \mathbb{E}_{(\cdot, \cdot)}\beta(\cdot, \cdot, x')v(x')$  is also bounded below: for all  $(x, a) \in \mathbf{G}$

$$\begin{aligned} \mathbb{E}_{x,a}\beta(x, a, X')v(X') &= \int_B \beta(x, a, x')v(x')P(x, a, dx') + \int_{\mathbf{X} \setminus B} \beta(x, a, x')v(x')P(x, a, dx') \\ &\geq \|\beta\|(e - m\|\kappa\|). \end{aligned}$$

Since  $Ev$  is bounded below and  $Ev/\kappa$  is bounded above, we conclude that  $\|Ev\|_\kappa < \infty$ , which implies  $Ev \in \mathcal{G}$  and then  $E\mathcal{V} \subset \mathcal{G}$ .

Next, we show that  $M\mathcal{G} \subset \mathcal{V}$ . Let  $g \in \mathcal{G}$ . Since  $r(x, \cdot)$  and  $g(x, \cdot)$  are u.s.c. on  $\Gamma(x)$  for all  $x \in \mathbf{X}$ , Proposition D.5 of [Hernández-Lerma and Lasserre \(2012a\)](#) implies that  $x \mapsto Mg(x) = \sup_{a \in \Gamma(x)} \{r(x, a) + g(x, a)\}$  is measurable. Moreover, Assumption 6.1 implies that for all  $x \in \mathbf{X}$

$$\begin{aligned} Mg(x) &= \sup_{a \in \Gamma(x)} \{r(x, a) + g(x, a)\} \leq \sup_{a \in \Gamma(x)} \{r(x, a)\} + \sup_{a \in \Gamma(x)} \{g(x, a)\} \\ &\leq d\kappa(x) + \|g\|_\kappa \kappa(x). \end{aligned}$$

Therefore,  $Mg/\kappa$  is bounded above. Also, since  $\|g\|_\kappa < \infty$  and  $\kappa$  is bounded, we have  $|g(\cdot)| \leq \|g\|_\kappa \kappa(\cdot) < \infty$  so that  $g$  is bounded below by some  $\underline{g} \in \mathbb{R}$ . Then, we have, for all  $(x, a) \in \mathbf{G}$ ,

$$\begin{aligned} \mathbb{E}_{x,a}Mg(X') &= \mathbb{E}_{x,a} \sup_{a' \in \Gamma(X')} \{r(X', a') + g(X', a')\} \geq \mathbb{E}_{x,a} \sup_{a' \in \Gamma(X')} \{r(X', a') + \underline{g}\} \\ &= \mathbb{E}_{x,a}\bar{r}(X') + \underline{g} = \hat{r}(x, a) + \underline{g}. \end{aligned}$$

Since  $\hat{r}$  is bounded below by assumption,  $\mathbb{E}_{(\cdot, \cdot)} Mg(X')$  is bounded below. Therefore, we have  $Mg \in \mathcal{V}$  and then  $M\mathcal{G} \subset \mathcal{V}$ . Now, since  $T = ME$  and  $R = EM$ , we obtain that  $T\mathcal{V} = ME\mathcal{V} \subset M\mathcal{G} \subset \mathcal{V}$  and  $R\mathcal{G} = EM\mathcal{G} \subset E\mathcal{V} \subset \mathcal{G}$ .  $\square$

**Lemma A.25.** *Let  $\mathcal{M}$  be regular and Assumption 6.1 hold. If  $\bar{g}$  is the unique fixed point of  $R$ , then  $\bar{v} = M\bar{g}$  is the unique fixed point of  $T$  and  $\bar{g} = E\bar{v}$ . In addition,  $R$  is globally stable on  $\mathcal{G}$  if and only if  $T$  is globally stable on  $\mathcal{V}$ .*

*Proof of Lemma A.25.* Let  $\mathcal{M}$  be regular and Assumption 6.1 hold. Let  $\bar{g}$  be unique the fixed point of  $R$ . Since  $T = ME$  and  $R = EM$ , we have  $M\bar{g} = MR\bar{g} = MEM\bar{g} = TM\bar{g}$ , so that  $M\bar{g}$  is a fixed point of  $T$ . Suppose that  $v \neq M\bar{g}$  is a fixed point of  $T$ . Then, since  $v$  is a fixed point of  $T$ , we obtain  $Ev = ETv = EMEv = REv$  implies that  $Ev$  is a fixed point of  $R$ . Since  $\bar{g}$  is the unique fixed point of  $R$ , we must have  $\bar{g} = Ev$  and then  $M\bar{g} = MEv = Tv = v$ . Therefore,  $M\bar{g}$  must be the unique fixed point of  $T$ .

The above statement also shows  $\bar{g} = E\bar{v}$ . For the second statement, observe the iteration  $T^n v = (ME)^n v = M(EM)^{n-1} Ev = MR^{n-1} Ev$  for any  $v \in \mathcal{V}$ . Since  $R$  is globally stable and  $Ev \in \mathcal{G}$ , we have  $R^{n-1} Ev \rightarrow \bar{g}$  as  $n \rightarrow \infty$ . Hence,  $T^n v \rightarrow M\bar{g}$  as  $n \rightarrow \infty$  for any  $v \in \mathcal{V}$ . Similarly, we can show that the global stability of  $T$  implies the global stability of  $R$ .  $\square$

**Lemma A.26.** *If  $\mathcal{M}$  is regular and Assumption 6.1 hold, then  $R$  is eventually contracting and globally stable on  $(\mathcal{G}, \|\cdot\|_\kappa)$ , and  $T$  is globally stable on  $(\mathcal{V}, \|\cdot\|)$ .*

*Proof of Lemma A.26.* Suppose that  $\mathcal{M}$  is regular and Assumption 6.1 hold. Lemma A.24 shows that  $R\mathcal{G} \subset \mathcal{G}$  and  $T\mathcal{V} \subset \mathcal{V}$ . Fix  $g, h \in \mathcal{G}$ . Assumption 6.1 (b) implies that

for all  $(x, a) \in \mathbf{G}$  we have

$$\begin{aligned}
|Rg(x, a) - Rh(x, a)| &= \left| \mathbb{E}_{x,a} \beta(x, a, X') \left[ \sup_{a' \in \Gamma(X')} \{r(X', a') + g(X', a')\} \right. \right. \\
&\quad \left. \left. - \sup_{a' \in \Gamma(X')} \{r(X', a') + h(X', a')\} \right] \right| \\
&\leq \mathbb{E}_{x,a} \left( \beta(x, a, X') \sup_{a' \in \Gamma(X')} |g(X', a') - h(X', a')| \right) \\
&\leq \mathbb{E}_{x,a} \beta(x, a, X') \|g - h\|_{\kappa} \kappa(X') \\
&\leq \|g - h\|_{\kappa} \kappa(x) \alpha L \mathbb{1}(x).
\end{aligned}$$

Then, iteration implies

$$\begin{aligned}
|R^2g(x, a) - R^2h(x, a)| &\leq \mathbb{E}_{x,a} \beta(x, a, X') \sup_{a' \in \Gamma(X')} |Rg(X', a') - Rh(X', a')| \\
&\leq \mathbb{E}_{x,a} \beta(x, a, X') \|g - h\|_{\kappa} \kappa(X') \alpha L \mathbb{1}(X') \\
&\leq \|g - h\|_{\kappa} \kappa(x) \alpha^2 L^2 \mathbb{1}(x).
\end{aligned}$$

Induction yields

$$|R^n g(x, a) - R^n h(x, a)| \leq \|g - h\|_{\kappa} \kappa(x) \alpha^n L^n \mathbb{1}(x)$$

for all  $(x, a) \in \mathbf{G}$  and  $n \in \mathbb{N}$ . Dividing  $\kappa(x)$  on the both sides and taking supremum, we obtain

$$\|R^n g - R^n h\|_{\kappa} \leq \|g - h\|_{\kappa} \alpha^n \|L^n \mathbb{1}\| \leq \|g - h\|_{\kappa} \alpha^n \|L^n\|.$$

Since there exists an  $n \in \mathbb{N}$  such that  $\|L^n\| < 1$  and  $\alpha < 1/\|L^n\|^{1/n}$ ,  $R$  is eventually contracting on  $(\mathcal{G}, \|\cdot\|_{\kappa})$ , whence it is also globally stable by the generalized Banach contraction mapping theorem. Finally, Lemma A.25 implies that  $T$  is globally stable.  $\square$

**Lemma A.27.** *If  $\mathcal{M}$  is regular and Assumption 6.1 hold, then a  $v$ -greedy policy and a  $g$ -greedy policy exist for all  $v \in \mathcal{V}$  and  $g \in \mathcal{G}$ .*

*Proof of Lemma A.27.* Let Condition A.1 and Assumption 6.1 hold. Let  $v \in \mathcal{V}$ . Condition A.1 and the proof of Lemma A.24 implies that  $a \mapsto r(x, a) + \mathbb{E}_{x,a} \beta(x, a, C') v(C')$  is u.s.c. for all  $x \in \mathbf{X}$ . Since  $\Gamma(x)$  is compact for all  $x \in \mathbf{X}$ , the  $v$ -greedy policy exists by the Maximum theorem. Let  $g \in \mathcal{G}$ . Then,  $a \mapsto r(x, a) + g(x, a)$  is u.s.c. for

all  $x \in \mathbf{X}$ , so the  $g$ -greedy policy exists by the Maximum theorem. The proof for Condition 6.1 is similar.  $\square$

*Proof of Lemma 6.1.* Let  $\mathcal{M}$  be regular and Assumption 6.1 hold. Lemma A.25 implies that  $\bar{v} = M\bar{g}$  is a unique fixed point of  $T$ , so  $\bar{v} = T\bar{v} \geq T_\sigma \bar{v}$  for all  $\sigma \in \Sigma$ . Fix  $\sigma \in \Sigma$ . Let  $\{X_t\}$  be a  $P_\sigma$ -Markov process and  $\beta_t^\sigma = \beta(X_t, \sigma(X_t), X_{t+1})$  for all  $t \in \mathbb{N}_0$ . Then, since  $\bar{v} \geq T_\sigma \bar{v}$  and  $\bar{g} = M\bar{v}$  by Lemma A.25, iteration implies that for all  $x \in \mathbf{X}$  and  $\sigma \in \Sigma$  we have

$$\begin{aligned}
\bar{v}(X_0) &\geq T_\sigma \bar{v}(X_0) = r(X_0, \sigma(X_0)) + \mathbb{E}_{X_0, \sigma(X_0)} \beta_0 \bar{v}(X_1) \\
&\geq r(X_0, \sigma(X_0)) + \mathbb{E}_{X_0, \sigma(X_0)} \beta_0^\sigma [r(X_1, \sigma(X_1)) + \mathbb{E}_{X_1, \sigma(X_1)} \beta_1^\sigma \bar{v}(X_2)] \\
&\geq \dots \\
&\geq \mathbb{E}_{X_0, \sigma(X_0)} \sum_{t=0}^N \prod_{i=0}^{t-1} \beta_i^\sigma r(X_t, \sigma(X_t)) + \mathbb{E}_{X_0, \sigma(X_0)} \beta_0^\sigma \beta_1^\sigma \cdots \beta_{N-1}^\sigma \mathbb{E}_{X_N, \sigma(X_N)} \beta_N^\sigma \bar{v}(X_{N+1}) \\
&= \mathbb{E}_{X_0, \sigma(X_0)} \sum_{t=0}^N \prod_{i=0}^{t-1} \beta_i^\sigma r(X_t, \sigma(X_t)) + \mathbb{E}_{X_0, \sigma(X_0)} \beta_0^\sigma \beta_1^\sigma \cdots \beta_{N-1}^\sigma \bar{g}(X_N, \sigma(X_N)).
\end{aligned} \tag{27}$$

Now, Assumption 6.1 implies

$$\begin{aligned}
|\mathbb{E}_{X_0, \sigma(X_0)} \beta_0^\sigma \beta_1^\sigma \cdots \beta_{N-1}^\sigma \bar{g}(X_N, \sigma(X_N))| &\leq \mathbb{E}_{X_0, \sigma(X_0)} \beta_0^\sigma \beta_1^\sigma \cdots \beta_{N-1}^\sigma |\bar{g}(X_N, \sigma(X_N))| \\
&\leq \mathbb{E}_{X_0, \sigma(X_0)} \beta_0^\sigma \beta_1^\sigma \cdots \mathbb{E}_{X_{N-1}, \sigma(X_{N-1})} \beta_{N-1}^\sigma \|g\|_\kappa \kappa(X_N) \\
&\leq \|g\|_\kappa \mathbb{E}_{X_0, \sigma(X_0)} \beta_0^\sigma \beta_1^\sigma \cdots \beta_{N-2}^\sigma \kappa(X_{N-1}) \alpha L \mathbb{1}(X_{N-1}) \\
&\leq \dots \\
&\leq \|\bar{g}\|_\kappa \kappa(X_0) \alpha^N L^N \mathbb{1}(X_0) \\
&\leq \|\bar{g}\|_\kappa \kappa(X_0) \|\alpha^N L^N \mathbb{1}\|.
\end{aligned}$$

Since there exists an  $n \in \mathbb{N}$  satisfying  $\alpha^n \|L\|^n < 1$ , letting  $N = pn + q$  with  $p, q \in \mathbb{N}_0$  and  $q < n$ , we obtain

$$\alpha^N \|L^N \mathbb{1}\| \leq (\alpha^{np} \|L^{np}\|) \alpha^q \|L^q\| \leq (\alpha^n \|L^n\|)^p \max_{q < n} \{\alpha^q \|L^q\|\} \rightarrow 0 \text{ as } n(N) \rightarrow \infty.$$

Therefore, letting  $N \rightarrow \infty$ , (27) and Lemma A.23 imply that  $\bar{v}(x_0) \geq v_\sigma(x_0)$  for all  $x_0$  and  $\sigma \in \Sigma$ . Hence, we have  $\bar{v} \geq \sup_\sigma v_\sigma = v^*$ . Next, since  $\bar{g} = E\bar{v}$  and  $\bar{g}$ -greedy



policy exists by Lemma A.27, there exists  $\sigma^*$  such that

$$\begin{aligned}\bar{v}(x) &= M\bar{g}(x) = r(x, \sigma^*(x)) + g(x, \sigma^*(x)) \\ &= r(x, \sigma^*(x)) + \mathbb{E}_{x, \sigma^*(x)} \beta(x, \sigma^*(x), x') \bar{v}(x') = T_{\sigma^*} \bar{v}(x).\end{aligned}\tag{28}$$

for all  $x \in \mathsf{X}$ . Hence, the same iteration of (27) with  $\bar{v} = T_{\sigma^*} \bar{v}$  implies  $\bar{v} = v_{\sigma^*} \leq v^*$ . We conclude that  $\bar{v} = v^*$ . The above arguments also show that a  $g^*$ -greedy policy  $\sigma^*$  is optimal:  $v^* = v_{\sigma^*}$  ((b)  $\Rightarrow$  (a)), and an optimal policy exists. We now show that part (a) implies (b). We write  $M_\sigma g(x) = r(x, \sigma(x)) + g(x, \sigma(x))$  for  $x \in \mathsf{X}$ ,  $g \in \mathcal{G}$  and  $\sigma \in \Sigma$ . Since  $\bar{g} = Ev^*$ , if  $\sigma^*$  is optimal, then we have

$$M_{\sigma^*} \bar{g} = M_{\sigma^*} Ev^* = T_{\sigma^*} v^* = T_{\sigma^*} v_{\sigma^*} = v_{\sigma^*} = v^* = Tv^* = MEv^* = M\bar{g}.$$

Hence,  $\sigma^*$  is  $\bar{g}$ -greedy. We next show that a policy is  $\bar{g}$ -greedy if and only if it is  $M\bar{g}$ -greedy. If  $\sigma \in \Sigma$  is  $\bar{g}$ -greedy:  $M_\sigma \bar{g} = M\bar{g}$ , then  $TM\bar{g} = MEM\bar{g} = MR\bar{g} = M\bar{g} = M_\sigma \bar{g} = M_\sigma R\bar{g} = M_\sigma EM\bar{g} = T_\sigma M\bar{g}$ , whence  $\sigma$  is  $M\bar{g}$ -greedy. Conversely, if  $\sigma$  is  $M\bar{g}$ -greedy:  $T_\sigma M\bar{g} = TM\bar{g}$ , then  $M_\sigma \bar{g} = M_\sigma R\bar{g} = M_\sigma EM\bar{g} = T_\sigma M\bar{g} = TM\bar{g} = MEM\bar{g} = MR\bar{g} = M\bar{g}$ , whence  $\sigma$  is  $\bar{g}$ -greedy. Similarly, the same method shows that a policy is  $v^*$ -greedy if and only if it is  $Ev^*$ -greedy.  $\square$

*Proof of Theorem 6.1.* Let Condition A.1 and Assumption 6.1 hold. Part (a) follows from A.23. Part (b) follows from Lemma A.26. Part (c), (d), (f), and (g) follow from Part (b), Lemma A.25, and 6.1. Part (e) follows from Part (b) and (c).  $\square$

*Proof of Lemma 6.2.* Let all the stated assumptions hold. We show the statement by induction. First, observe that

$$\begin{aligned}\hat{r}(x, a) &= \mathbb{E}_z u(R(\varepsilon')(w - c) + y(\varepsilon', Z')) \geq \mathbb{E}_z u(y(Z', \varepsilon')) > -\infty, \\ \bar{r}(x) &= \sup_{0 \leq c \leq w} u(c) = u(w) \leq pw + q =: \kappa(x).\end{aligned}$$

The last equation implies that we can set  $d = 1$  in Assumption 6.1. Also, observe that

$$\mathbb{E}_{x,a} \kappa(X') = p\mathbb{E}_z (R(\varepsilon')(w - c) + y(Z', \varepsilon')) + q$$

is continuous in  $(x, a) = (w, z, c)$ . Therefore, Assumption 6.1 (a) and (c) hold. Next, note that

$$\begin{aligned} \frac{\mathbb{E}_{x,a}\beta(x, a, X')\kappa(X')}{\kappa(x)} &= \frac{\mathbb{E}_z\beta(z, Z')(pW' + q)}{pw + q} \\ &= \frac{\mathbb{E}_z\beta(z, Z')[p(R(\varepsilon')(w - c) + y(Z', \varepsilon')) + q]}{pw + q} \\ &\leq \frac{\mathbb{E}_z\beta(z, Z')[p(R(\varepsilon')w + y(Z', \varepsilon')) + q]}{pw + q}. \end{aligned}$$

Since the right-hand side is a monotone function of  $w$  and then achieves the supremum at either  $w = 0$  or  $w = \infty$ , we have

$$\frac{\mathbb{E}_{x,a}\beta(x, a, X')\kappa(x')}{\kappa(x)} \leq \max \left\{ \frac{\mathbb{E}_z\beta(z, Z')[py(Z', \varepsilon') + q]}{q}, \mathbb{E}_z\beta(z, Z')\mathbb{E}R(\varepsilon') \right\}.$$

where the last inequality follows from that  $\varepsilon_t$  is IID. Then, since  $q > 1$  can be arbitrarily large, (19) implies

$$\frac{\mathbb{E}_z\beta(z, Z')[py(Z', \varepsilon') + q]}{q} \rightarrow \mathbb{E}_z\beta(z, Z')$$

as  $q \rightarrow \infty$ . Since  $\mathbb{E}R(\varepsilon') > 1$ , we have

$$\frac{\mathbb{E}_{x,a}\beta(x, a, X')\kappa(X')}{\kappa(x)} \leq \mathbb{E}_z\beta(z, Z')\mathbb{E}R(\varepsilon') =: L\mathbb{1}(x)$$

for large enough  $q > 1$ . Hence, we have  $\mathbb{E}_{x,a}\beta(x, a, X')\kappa(X')L^0\mathbb{1}(x') \leq \kappa(x)L\mathbb{1}(x)$  for large enough  $q > 1$ . Assume the induction hypothesis that  $\mathbb{E}_{x,a}\beta(x, a, X')\kappa(X')L^n\mathbb{1}(X') \leq \kappa(x)L^{n+1}\mathbb{1}(x)$  for some  $n \in \{1, \dots, m-1\}$ . Then, iteration yields

$$\begin{aligned} &\frac{\mathbb{E}_{x,a}\beta(x, a, X')\kappa(x')L^{n+1}\mathbb{1}(X')}{\kappa(x)} \\ &\leq \frac{\mathbb{E}_z\beta(z, Z')[p(R(\varepsilon')w + y(Z', \varepsilon')) + q]\mathbb{E}_{Z'}\beta_1\beta_2 \dots \beta_{n+1}(\mathbb{E}R(\varepsilon'))^{n+1}}{pw + q} \\ &\leq \max \left\{ \frac{\mathbb{E}_z\beta(z, Z')[py(Z', \varepsilon') + q]\mathbb{E}_{Z'}\beta_1\beta_2 \dots \beta_{n+1}(\mathbb{E}R(\varepsilon'))^{n+1}}{q}, \right. \\ &\quad \left. \mathbb{E}_z\beta(z, Z')\beta_1\beta_2 \dots \beta_{n+1}(\mathbb{E}R(\varepsilon'))^{n+2} \right\} \\ &\rightarrow \max \{ \mathbb{E}_z\beta(z, Z')\beta_1\beta_2 \dots \beta_{n+1}(\mathbb{E}R(\varepsilon'))^{n+1}, \\ &\quad \mathbb{E}_z\beta(z, Z')\beta_1\beta_2 \dots \beta_{n+1}(\mathbb{E}R(\varepsilon'))^{n+2} \} \\ &\leq L^{n+2}\mathbb{1}(x), \end{aligned}$$

where the first inequality uses the definitions of  $\kappa$  and  $L^{n+1}\mathbb{1}$ , the second inequality follows from that the supremum is at either  $w = 0$  or  $w = \infty$ , the limit follows from taking  $q$  arbitrarily large, and the last inequality uses  $\mathbb{E}R(\varepsilon') > 1$ . Therefore, induction implies that there is large enough  $q(n) > 1$  such that

$$\mathbb{E}_{x,a}\beta(x, a, X')\kappa(X')L^n\mathbb{1}(X') \leq \kappa(x)L^{n+1}\mathbb{1}(x)$$

for all  $x \in \mathbf{X}$  and all  $n \in \{1, \dots, m-1\}$ .  $\square$

#### A.5. Proofs in Section 7.

*Proof of Theorem 7.1.* Let Condition 7.1, Assumption 7.1, 7.2, and 7.3 hold. Part (a) follows from 7.1 and 7.2. Part (b), (c), (e), and (f) of optimality follow from the proof of Theorem 5.3 of Ma et al. (2022) and Lemma 7.2 and 7.3. Part (d) follows from (a) and (b).  $\square$

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